What is light?

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Abstract

The purpose of this paper is to answer the question, 'What is light?', from a mathematical and foundational perspective. The paper begins by exploring the relevance of space-time symmetries, and the nature of polarization, before a detailed exposition of the quantized radiation field, and the difficulties created by gauge freedom.

Attention then turns to the interaction of light with matter, beginning with the coupled electromagnetic field, then progressing to the representation of scattering and virtual particles in quantum electrodynamics. This provokes an analysis of the Coulomb electrostatic field, and the question of whether longitudinal and scalar photons exist. It is argued that the longitudinal component of the electric field is associated with the state-space of a charged matter field, not the Fock space of the free electromagnetic field, hence the presence of electrostatic fields is consistent with the vacuum state of the free electromagnetic field. It is also argued that scalar and longitudinal photons do indeed exist as links in the spacelike networks into which the Coulomb interaction can be decomposed.

Consideration of the stimulated emission of light leads to a general exposition and analysis of the 'coherent states' of the quantized radiation field. As a by-product of this, a novel explanation is proposed for why there is something classical rather than nothing classical. An attempt is made to develop this into a fully-fledged universe creation scenario. The role of fermions and the gravitational degrees of freedom in such a scenario are discussed, and a comparison is drawn with the inflationary cosmological scenario.

The role of coherent states in our concept of the classical world is then critically analysed. The notion that the classical states of the radiation field are emergent from the quantum states is rejected. In particular, it is argued that the classical states of light do not emerge in the limit where there are large numbers of photons, and it is pointed out that the putative emergent classical states fail the test of reference-frame independence.

The paper concludes by expounding the implications of the nature of light for 'decoherence', a way of trying to reconcile quantum theory with the apparent nature of the macroscopic world.

Contents

1 Introduction

A plurality of mathematical structures exist for the representation of light: it is represented as bundles, pencils and beams of 1-dimensional rays in geometrical optics; as a tensor field satisfying a wave-equation in Maxwellian electromagnetism; and as the elements of a photonic Fock space in quantum field theory. All of which begs the question, 'What is light?' The purpose of this paper is to answer that question from a mathematical and foundational perspective.

The second section begins with a concise exposition of light as a classical electromagnetic wave. Noting that such waves can be decomposed into different modes and polarizations, prompts the question 'What is polarization?' To answer this, we digress to consider light as a representation of the space-time symmetry group. Equipped with this perspective, we then consider the detailed mathematical representation of polarization.

The third section of the paper begins with a brief recapitulation of the mathematical structure of Fock space and the quantum vacuum. In particular, the isomorphism between Fock space and the space of wavefunctionals on the one-particle space is made explicit. The remainder of the section is devoted to the quantized radiation field, and the problems created by gauge freedom.

The fourth section considers how interactions between light and matter are represented, beginning with the coupled electromagnetic field in the classical theory, and progressing to the representation of interacting fields in quantum field theory. This provokes an analysis of the Coulomb electrostatic field, and the question of whether longitudinal and scalar photons exist. As a special case of the interaction between light and matter, the spontaneous emission of light is explained. This is then contrasted with the production of laser light by stimulated emission.

Laser light is represented mathematically by coherent states in photonic Fock space, and the fifth section is devoted to an exposition and analysis of these privileged states of the radiation field.

This analysis leads to a number of important philosophical points. Section six is devoted to the first of these, a possible explanation for why there is something classical rather than nothing classical. An attempt is made to develop this into a fully-fledged universe creation scenario. The role of fermions and the gravitational degrees of freedom in such a scenario are discussed, and a comparison is drawn with the inflationary cosmological scenario.

The seventh section provides a general discussion of the implications of coherent states for the concept of classicality. A number of the common beliefs, in particular the notion that the classical world emerges from the quantum world, are shown to be false.

The paper concludes with a section on the implications of the nature of light for 'decoherence', a way of trying to reconcile quantum theory with the apparent nature of the macroscopic world.

2 What is light?

2.1 Classical electromagnetic waves

According to classical electromagnetism, light is a solution of the free-field Maxwell equations. Under a Lorentz choice of gauge, the electromagnetic potential is mapped to a real vector potential A satisfying $\nabla \cdot A = 0$. Given a real vector potential A, the electromagnetic field strength is $F = dA$. The Maxwell equations for a vector potential obtained with a Lorentz choice of gauge are simply:

$$\nabla \cdot dA = 0, \quad \nabla \cdot A = 0 \, .$$

These equations are equivalent to, (Derdzinski 2002, Appendix 48):

$$\Box A = 0, \quad \nabla \cdot A = 0 \, .$$

The first equation here is the wave equation on Minkowski space-time $\mathcal{M}$,

$$(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2})A = \Box A = (\Box A_\mu)dx^\mu = 0 \, ,$$

where $\Box$ is the d'Alembertian.

Such solutions can be subjected to a two-fold decomposition: (i) a Fourier decomposition into monochromatic modes of different wavelengths; and (ii) a decomposition of monochromatic light into different polarizations.

Both of these decompositions were part of classical optics, theoretically and in terms of the historical experiments which explored their nature. Yet whilst the Fourier decomposition is clearly classical, the polarization decomposition introduces a quantum-mechanical substructure into classical theory. Moreover, whilst numerous physicists are fond of claiming that there is no analogue of quantum spin in classical physics, the case of polarization provides a counter-example to such claims. To gain a better understanding of what polarization is, the next subsection considers the role of space-time symmetries.

2.2 Symmetry

Wigner used the method of induced group representation to obtain an irreducible unitary Hilbert space representation of the universal cover of the local space-time symmetry group for every type of elementary particle. The largest group of space-time symmetries satisfied by all the elementary particles is the *restricted* Poincaré group, $SO_0(3,1) \ltimes \mathbb{R}^{3,1}$. This group does not contain time reversal operations, parity transformations, or combinations thereof. The universal cover of the *restricted* Poincaré group is $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{3,1}$, and the irreducible unitary representations of this group can be classified by two parameters, (m, s), mass and spin, hence each elementary particle is associated with a Hilbert space $\mathscr{H}_{m,s}$.

The photon is a particle of mass 0 and spin 1. The Wigner approach obtains a Hilbert space $\mathscr{H}_{0,1}$ of square-integrable cross-sections of a vector bundle $E^+_{0,s}$

over the forward light cone $\mathscr{V}_0^+$ in energy-momentum space, with typical fibre isomorphic to $\mathbb{C}^1$.

However, the electromagnetic force respects symmetries such as parity transformations. As a consequence, it is conventional to treat the interaction carrier of the electromagnetic force, the photon, as an irreducible representation of $O^\uparrow(3,1) \ltimes \mathbb{R}^{3,1}$, the isochronous Poincaré group, (also called the orthochronous Poincaré group). This group consists of both the identity component of the Poincaré group, and the component which contains the operation of parity reversal, $\mathscr{P} : (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, -x_2, -x_3)$.

Whilst the irreducible representations of the restricted Poincaré group are parameterized by $s \in \frac{1}{2}\mathbb{Z}$, the irreducible representations of the isochronous Poincaré group are parameterized by $t \in \frac{1}{2}\mathbb{Z}_+$. For $t \neq 0$, the t-representation is a space of cross-sections of a bundle over the forward light cone with typical fibre isomorphic to $\mathbb{C}^2$. It can be decomposed into a direct sum of the $s = t$ and $s = -t$ representations. In the case of a mass zero particle, the discrete parameter s is often alternatively referred to as the 'helicity' or the 'polarization' of the particle, and the spin is treated as the absolute value $|s|$.

Thus, to summarise, the photon is treated as a particle of mass $m = 0$ and spin $t = 1$. The bundle $E^+_{0,t} = E^+_{0,1}$ possesses sub-bundles of helicity $s = 1$ and $s = -1$, which correspond to the right-handed and left-handed 'circular polarization' of a photon. These sub-bundles correspond to the $E^+_{m,s} = E^+_{0,1}$ and $E^+_{m,s} = E^+_{0,-1}$ bundles used in the representations of the restricted Poincaré group.

With an understanding of how polarization enters the fray in terms of spacetime symmetries, in the next subsection we proceed to explore the nature of polarization in greater depth.

2.3 Polarization

Explanations of the polarization of light typically begin by introducing 'linear' (or 'plane') polarization. They then proceed to define two states of opposite 'circular polarization' as superpositions of orthogonal states of linear polarization. The most general polarized state is said to be one of 'elliptical polarization', which can be defined as an arbitrary superposition of the states of opposite circular polarization.

The states of definite helicity correspond to the states of circular polarization. A general state of polarization, (elliptical polarization) will be one in which the helicity is indefinite. Hence, a general state will belong to the direct sum of two irreducible representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{3,1}$.

The discovery of polarization pre-dated the discovery of quantum theory, (see Guillemin and Sternberg, 1984). Hence, the polarization of light can be represented in both geometrical optics, and Maxwellian electromagnetism, as well as quantum theory. The possible polarization states of a ray of light, a free electromagnetic wave, or a photon, are represented by a 2-dimensional complex vector space isomorphic to $\mathbb{C}^2$. In the case of a wave, this polarization space is tensored with the space of wave solutions. The real component of these complex

solutions corresponds to the electric field vector of a classical electromagnetic wave.

In the case of a plane-wave with a spacelike wave-vector $\mathbf{k}$, the electric field vector is constrained to lie in the real 2-dimensional space perpendicular to $\mathbf{k}$. The complexification of this space corresponds to the space of complex polarization vectors.

For example, start with a pair $(\mathbf{e}_1, \mathbf{e}_2)$ of spacelike orthogonal unit vectors which span the spacelike plane perpendicular to $\mathbf{k}$. If we align the x-axis with $\mathbf{e}_1$, the y-axis with $\mathbf{e}_2$, and the z-axis with $\mathbf{k}$, then $\mathbf{e}_1$ represents light which is linearly polarized along the x-axis, and $\mathbf{e}_2$ represents light which is linearly polarized along the y-axis. We can therefore denote our basis as $(\mathbf{e}_x, \mathbf{e}_y)$. In terms of the electric field vector, a state of linear polarization along an axis $\{a\mathbf{e} : a \in \mathbb{R}^1\}$ is one in which the tip of the electric vector oscillates back and forth along the line spanned by $\mathbf{e}$.

The two-dimensional *real* vector-space generated by $(\mathbf{e}_x, \mathbf{e}_y)$ captures all the possible states of *linear* polarization. To be precise, the set of all possible states of linear polarization corresponds to the real projective space $\mathbb{RP}^1$ defined by all the 1-dimensional subspaces through the origin of this 2-dimensional real vector space.

Linear polarization states exist in any given direction within the real plane perpendicular to $\mathbf{k}$. Given the basis $(\mathbf{e}_x, \mathbf{e}_y)$, one can define an alternative basis $(\mathbf{e}_{x'}, \mathbf{e}_{y'})$ which is rotated by an angle θ from the original pair:

$$\begin{aligned} \mathbf{e}_x &= \cos\theta \mathbf{e}_{x'} + \sin\theta \mathbf{e}_{y'} \\ \mathbf{e}_y &= -\sin\theta \mathbf{e}_{x'} + \cos\theta \mathbf{e}_{y'} \end{aligned} .$$

If a beam of photons linearly polarized along the $\mathbf{e}_x$-axis impinges on a Polaroid sheet which is oriented along the $\mathbf{e}_{x'}$-axis, then only a fraction $\cos^2\theta$ will be transmitted. In effect, we have the familiar quantum-mechanical scenario where a system has been prepared into a state which is not an eigenstate of the quantity being measured, and a decomposition of the state vector in terms of its components along the eigenvectors of the measured quantity, $\Psi = c_1\psi_1 + c_2\psi_2 = \mathbf{e}_x = \cos\theta \mathbf{e}_{x'} + \sin\theta \mathbf{e}_{y'}$. The probability of measuring polarization along the $\mathbf{e}_{x'}$-axis is the square modulus of the coefficient $c_1 = \cos\theta$, which in this case happens to be a real coefficient.

The states of linear polarization are clearly not classical. The photons in a beam polarized along the $\mathbf{e}_x$-axis are indefinite with respect to $(\mathbf{e}_{x'}, \mathbf{e}_{y'})$ until a measurement-like interaction takes place. All the familiar logic of quantum theory can be found in the states of linear polarization, despite the fact that it is only a real space.

However, whilst the space of linearly polarized states is non-classical, it does not comprise the entire space of polarized states. This is obtained from the complexification of the space of linearly polarized states. Define a basis $(\mathbf{e}_+, \mathbf{e}_-)$ of this complex space as follows:

$$\begin{aligned} \mathbf{e}_+ &= \mathbf{e}_x + i\mathbf{e}_y \\ \mathbf{e}_- &= \mathbf{e}_x - i\mathbf{e}_y \end{aligned} \ .$$

This is the helicity basis, or equivalently, the eigenstates of circular polarization. $\mathbf{e}_+$ is the state of positive helicity, or left-circular polarization, while $\mathbf{e}_-$ is the state of negative helicity, or right-circular polarization. In a circularly polarized state, the tip of the electric field vector will sweep out a circle in the plane perpendicular to $\mathbf{k}$.

A general polarized state is then expressible as a superposition of the positive and negative helicity eigenstates:

$$\Psi = c_+\mathbf{e}_+ + c_-\mathbf{e}_- \ .$$

These are the elliptically polarized states. In general, the tip of the electric field vector will sweep out an ellipse in the plane perpendicular to $\mathbf{k}$. Both circular polarization and linear polarization can be seen as limiting cases of elliptical polarization.[1]

However, the helicity eigenstates provide only one possible basis for the space of polarization vectors. Orthogonal linear polarization eigenstates provide alternative bases for the entire space. Whilst the real vector space generated by a pair of linear polarization eigenstates only spans the linear polarization subspace, the set of *complex* linear combinations spans the entire space.

There is one particularly elegant way of representing the space of polarization states. The space of states corresponds to the set of complex 1-dimensional subspaces of a 2-dimensional complex vector space. This is the complex projective line $\mathbb{CP}^1$, the Riemann sphere,[2] which can be coordinatized as the set of ratios of complex number pairs, w/z.

In general, the ratio of a pair of complex numbers is determined by the relative size of the respective amplitudes, and by the phase-difference:

$$\frac{w}{z} = \frac{r_w e^{i\theta_w}}{r_z e^{i\theta_z}} = \frac{r_w}{r_z} e^{i(\theta_w - \theta_z)} \ .$$

Orthogonal states correspond to antipodal points on the Riemann sphere. If the positive helicity eigenstate $\Psi = \mathbf{e}_+$ is mapped to the North pole, and the ratio $c_-/c_+ = 0/1 = 0$, then the negative helicity eigenstate $\Psi = \mathbf{e}_-$ is mapped to the South pole and the ratio $c_-/c_+ = 1/0 = \infty$.

If the helicity eigenstates are mapped to the poles, the set of linear polarization states is mapped to the equator. For example,

$$\frac{1}{2}\mathbf{e}_+ + \frac{1}{2}\mathbf{e}_- = \frac{1}{2}(\mathbf{e}_x + i\mathbf{e}_y) + \frac{1}{2}(\mathbf{e}_x - i\mathbf{e}_y) = \mathbf{e}_x \ ,$$

and

[1] As the ratio of semi-major axis and semi-minor axis of an ellipse tends to infinity, it flattens into a line oriented in a particular direction, corresponding to the case of linear polarization.

[2] The optics literature also refers to this geometrical representation as the Poincaré sphere.

$$-i\frac{1}{2}\mathbf{e}_+ + i\frac{1}{2}\mathbf{e}_- = -i\frac{1}{2}(\mathbf{e}_x + i\mathbf{e}_y) + i\frac{1}{2}(\mathbf{e}_x - i\mathbf{e}_y) = \mathbf{e}_y \ .$$

In terms of ratios of numbers, the first case $\mathbf{e}_x$ corresponds to $c_-/c_+ = 1$, and the orthogonal state on the equator $\mathbf{e}_y$ corresponds to $c_-/c_+ = -1$, two complex numbers with unit modulus and a phase difference of π.

States of elliptical polarization lie between the poles and the equator. If the state of left-circular polarization corresponds to the North pole, then left-elliptical states lie in the Northern hemisphere, and right-elliptical states lie in the Southern hemisphere. Points on lines of constant latitude correspond to ellipses of constant ellipticity, but a varying angle of the semi-major axis. Points on lines of constant latitude correspond to a constant angle of the semi-major axis, but a varying ellipticity.

To link the polarization states with the electric field component of a classical free electromagnetic wave, the polarization state $\Psi = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$ can be inserted as the complex amplitude of a plane-wave:

$$\Psi e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \ .$$

The electric field vector is then deemed to be the real component of this complex field:

$$\mathbf{E}(\mathbf{r}, t) = Re[\Psi e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \ .$$

For example, if we start with a basis of linear polarization eigenstates $(\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{e}_x, \mathbf{e}_y)$, and take the positive helicity state $\mathbf{e}_+ = \mathbf{e}_x + i\mathbf{e}_y$, then:

$$\begin{aligned} Re[E_0(\mathbf{e}_x + i\mathbf{e}_y)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] &= E_0\mathbf{e}_x\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + E_0\mathbf{e}_y\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \pi/2) \\ &= E_0\mathbf{e}_x\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) - E_0\mathbf{e}_y\sin(\mathbf{k}\cdot\mathbf{r} - \omega t) \ . \end{aligned}$$

For any point $\mathbf{r}$ in the plane perpendicular to $\mathbf{k}$, this defines the clockwise rotation of a vector around a circle. E_0 is the real magnitude of this polarization vector. The clockwise rotation corresponds to positive helicity. Taking the negative helicity polarization state, $\mathbf{e}_- = \mathbf{e}_x - i\mathbf{e}_y$, then:

$$\begin{aligned} Re[E_0(\mathbf{e}_x - i\mathbf{e}_y)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] &= E_0\mathbf{e}_x\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + E_0\mathbf{e}_y\cos(\mathbf{k}\cdot\mathbf{r} - \omega t - \pi/2) \\ &= E_0\mathbf{e}_x\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + E_0\mathbf{e}_y\sin(\mathbf{k}\cdot\mathbf{r} - \omega t) \ . \end{aligned}$$

This defines the anti-clockwise rotation of a vector of length E_0 around a circle, which corresponds to negative helicity.

To close this section on polarization, note that a space of mixed states can be built from the pure polarized states defined above. Whilst the pure states correspond to points on the surface of the Riemann sphere, the mixed states correspond to points in the interior of the unit ball enclosed by the Riemann sphere. These mixed states can be represented as density operators such as the following:

$$D = a_1 P_{\mathbf{e}_1} + a_2 P_{\mathbf{e}_2} \ ,$$

where $P_{\mathbf{e}_i}$ is the projection operator onto the subspace spanned by the vector $\mathbf{e}_i$, and the coefficients define a convex linear combination of projection operators, so that $a_1 + a_2 = 1$, and $a_i \geq 0$.

In the case where $a_1 = a_2$, the mixed state is referred to as representing 'unpolarized' or 'totally unpolarized' light. This corresponds to the point at the centre of the unit ball enclosed by the Riemann sphere. It can be represented as the equally weighted convex sum of *any* pair of orthogonal pure states at either end of a straight line passing through the centre. For example,

$$0.5P_{\mathbf{e}_+} + 0.5P_{\mathbf{e}_-} = 0.5P_{\mathbf{e}_x} + 0.5P_{\mathbf{e}_y} \ .$$

Direct sunlight, or sunlight which has undergone numerous scattering interactions passing through cloud, is considered to be unpolarized in this sense.

A general convex sum of pure polarization states is said to represent 'partially polarized' light. The 'degree of polarization' p can be defined as an excess of one pure polarization state over its orthogonal state. Thus, if

$$D = 0.25P_{\mathbf{e}_+} + 0.75P_{\mathbf{e}_-} \ ,$$

then the degree of polarization is the contrast,

$$p = \frac{|0.25 - 0.75|}{0.25 + 0.75} = 0.5 \ .$$

This mixed state would be represented a point on the diameter between the North pole and South pole of the Riemann sphere. If the state of right-circular polarization $\mathbf{e}_-$ is mapped to the South pole, then this mixed state would be half-way between the centre and the South pole.[3]

In operational terms the state of polarization of a light-beam is characterised by four numbers $(S_0, \mathbf{S}) = (S_0, S_1, S_2, S_3)$, called the Stokes parameters. These can be measured by the fraction of light transmitted through a sequence of four filters. S_0, measured by the first filter, is simply the intensity of the light, and is used to normalize the other three parameters. $(S_1/S_0, S_2/S_0, S_3/S_0)$ correspond to the three Cartesian coordinates of the 'Stokes vector'. This vector lies on or within the unit ball enclosed by the Riemann sphere.

If one chooses the second filter to be a linear polarizer with a horizontal transmission axis, then S_1/S_0 provides a coordinate along the line passing through the horizontal $\mathbf{e}_x$ and vertical $\mathbf{e}_y$ states of linear polarization, antipodal to each other on the equator of the Riemann sphere. If one chooses the second filter to be a linear polarizer with a transmission axis at 45° to the horizontal, then S_2/S_0 provides a coordinate along the line passing through orthogonal states of linear polarization which are rotated by $\pi/2$ around the equator from the first pair of linear polarization states. If the third filter is chosen to be one which only transmits right-circularly polarized light, then S_3/S_0 provides a coordinate along the line passing through North and South poles.

[3]Personal communication with Aaron Goldberg.

In terms of this Stokes vector, the degree of polarization is, in general, given by:

$$p = \frac{||\mathbf{S}||}{S_0} = \frac{\sqrt{(S_1^2 + S_2^2 + S_3^2)}}{S_0} \ .$$

The sunlight scattered from atmospheric molecules in a clear sky is partially polarized, with the degree of polarization peaking in a circular band at an angular separation of 90° from the position of the Sun. Thus, when the Sun is at the zenith at mid-Summer, the band of maximum polarization coincides with the 360 degrees of the horizon, where the light is horizontally polarized. Conversely, during sunrise or sunset at the vernal or autumnal equinox, the band of maximum polarization coincides with the North-South meridian, the polarization axis being vertical where the meridian meets the horizon.

One might be tempted to interpret a mixed state $D = a_1 P_{\mathbf{e}_1} + a_2 P_{\mathbf{e}_2}$ as representing a population of photons in which a fraction a_1 possess a polarization state of $\mathbf{e}_1$, and the remaining fraction a_2 possess the polarization state $\mathbf{e}_2$. The probabilities might appear to express incomplete knowledge rather than intrinsic indefiniteness. However, density operators which are not themselves projection operators have an infinite number of possible decompositions. In particular, if the unpolarized state of light is expressed as a mixture of linear polarization states,

$$D = 0.5 P_{\mathbf{e}_x} + 0.5 P_{\mathbf{e}_y} \ ,$$

it can also be expressed as a mixture of circular polarization states:

$$D = 0.5 P_{\mathbf{e}_+} + 0.5 P_{\mathbf{e}_-} \ .$$

Hence, the probabilities corresponding to the coefficients cannot be interpreted as expressing merely incomplete knowledge.

So much for polarization. In the next section we turn to quantum field theory, and then the quantization of the radiation field, expressed as a superposition of various plane-wave modes and polarizations.

3 Quantum Field Theory

3.1 Fock space

According to modern mathematical physics, the physical world is composed of quantum fields, and particles are merely 'excitations' of those fields. There are two types of such quantum fields: matter fields and gauge force fields. All quantum theories are obtained by applying quantization algorithms to classical theories, and quantum field theory is the upshot of two levels of quantization: in first-quantized relativistic quantum theory, each type of matter field or gauge field corresponds to a Hilbert space of cross-sections of a vector bundle over space-time, satisfying certain conditions; in second-quantized relativistic quantum field theory (quantum field theory proper), Fock spaces are constructed from these first-quantized particle spaces. In the guise of 'one-particle states', these vector bundle cross-sections are, more or less, the types of thing which are created or annihilated in second-quantized relativistic quantum field theory (McCabe 2007).

Wigner's classification of the projective, unitary, irreducible representations of the Poincaré group supplies an infinite dimensional Hilbert space for a particle with any possible combination of mass m and spin/helicity s.[4] These Hilbert spaces are the so-called single-particle Hilbert spaces from which the Fock spaces of the second-quantized theory can be built. Given the single-particle Hilbert space $\mathscr{H}_{m,s}$ for a bosonic system, the Fock space is

$$\mathscr{F}_{m,s} = \mathscr{F}(\mathscr{H}_{m,s}) = \bigoplus_{n=0}^{\infty} \mathscr{H}_{m,s}^{\odot_n} \ ,$$

where $\mathscr{H}_{m,s}^{\odot_n}$ is the n-fold symmetric tensor product of $\mathscr{H}_{m,s}$. The symmetric tensor product is the image of the tensor product under the following projection mapping:

$$S_n^+(f_1 \otimes \cdots \otimes f_n) = (n!)^{-1/2} \sum_{\sigma} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \ .$$

The sum here is over all the permutations σ of the indices $(1, 2, ..., n)$. The symmetric tensor product is $\mathscr{H}_{m,s}^{\odot_n} = S_n^+(\mathscr{H}_{m,s}^{\otimes_n})$.

Given the single-particle Hilbert space $\mathscr{H}_{m,s}$ for a fermionic system, the Fock space is

$$\mathscr{F}_{m,s} = \mathscr{F}(\mathscr{H}_{m,s}) = \bigoplus_{n=0}^{\infty} \mathscr{H}_{m,s}^{\wedge_n} \ ,$$

where $\mathscr{H}_{m,s}^{\wedge_n}$ is the n-fold anti-symmetric tensor product of $\mathscr{H}_{m,s}$. The anti-symmetric tensor product is the image of the tensor product under the following projection mapping:

[4] The projective, unitary, irreducible representations of the restricted Poincaré group are in bijective correspondence with the ordinary, unitary, irreducible representations of its universal covering group, $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{3,1}$.

$$S_n^-(f_1 \otimes \cdots \otimes f_n) = (n!)^{-1/2} \sum_\sigma \chi(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \ ,$$

where $\chi(\sigma)$ is the sign of the permutation. Each permutation of a set of elements can be expressed as a sequence of transpositions of pairs of elements, and, in this sense, $\chi(\sigma) = 1$ for permutations obtained from an equal number of transpositions, and $\chi(\sigma) = -1$ for permutations obtained from an odd number of transpositions. The anti-symmetric tensor product is $\mathscr{H}_{m,s}^{\wedge_n} = S_n^-(\mathscr{H}_{m,s}^{\otimes_n})$.

There is an alternative, albeit mathematically equivalent approach, in which the quantum field state space is obtained as a space K of complex-valued polynomial functionals defined upon the complex infinite-dimensional single-particle space. In this approach, the complement of the degree $n-1$ polynomials in the space of degree n polynomials corresponds to the n-particle subspace of Fock space, (Derdzinski 2002, Section 3.6). This approach treats states in quantum field theory as 'wavefunctionals' $\phi(f)$, describing superpositions of different (complexified) classical field configurations f.

The isomorphism between these two representations uses the fact that for a collection of Hilbert spaces $\mathscr{H}_1, \ldots, \mathscr{H}_n$, equipped with inner products $\langle \cdot, \cdot \rangle_1, \ldots, \langle \cdot, \cdot \rangle_n$, the inner product $\langle \cdot, \cdot \rangle$ on the tensor product $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$ is defined by

$$\langle f, g \rangle = \prod_{k=1}^n \langle f_k, g_k \rangle_k = \langle f_1 \ , g_1 \rangle_1 \cdots \langle f_n, g_n \rangle_n$$

for simple tensors,

$$f = f_1 \otimes \cdots \otimes f_n \ , \quad g = g_1 \otimes \cdots \otimes g_n \ .$$

This extends to linear combinations of such simple tensors using the bilinearity of the inner product. (Hereafter, we will drop the subscripts on inner products to reduce the notational clutter.)

A degree $p = p_1 + \cdots + p_k$ monomial $\phi(w_1, \ldots, w_n) = w_1^{p_1} \cdots w_k^{p_k}$ on $\mathbb{C}^k$ can be transplanted to a degree p monomial $\phi(f)$ on $\mathscr{H}_{m,s}$ by means of k vectors in the 1-particle Hilbert space, $\{e_j \in \mathscr{H}_{m,s} : j = 1, \ldots, k\}$, such that:

$$\phi(f) = \langle e_1, f \rangle^{p_1} \cdots \langle e_k, f \rangle^{p_k} \ .$$

If the e_j are selected from an orthonormal basis of $\mathscr{H}_{m,s}$, then, (with appropriate normalizing factors), the set of such monomials provide an orthonormal basis for K, the space of polynomial functions on the 1-particle Hilbert space.

The selection of an orthonormal basis e_j of the 1-particle Hilbert space determines an expansion of each vector $f \in \mathscr{H}_{m,s}$:

$$f = \sum_{j=1}^\infty c_j e_j = \langle e_j, f \rangle e_j \ .$$

Hence, the polynomials on $\mathbb{C}^k$ can be transplanted to polynomials on $\mathscr{H}_{m,s}$ by applying the monomial exponents to the expansion coefficients in an orthonormal decomposition of each vector.

The elements of the n-particle subspace $\mathscr{H}_{m,s}^{\otimes_n}$ map to n-degree monomials. For example, suppose we have a third-degree monomial ϕ with $p_1 = 1, p_4 = 2$. Then,

$$\phi(f) = \langle e_1, f\rangle\langle e_4, f\rangle^2 = \langle e_1 \otimes e_4 \otimes e_4, f \otimes f \otimes f\rangle \ .$$

The third-degree monomial maps to the element $e_1 \otimes e_4 \otimes e_4$ in the 3-particle Hilbert space. Conversely, a linear combination of vectors in the 3-particle Hilbert space maps to a sum of third-degree monomials on the 1-particle Hilbert space. For example, $\Psi = c_{144}e_1 \otimes e_4 \otimes e_4 + c_{444}e_4 \otimes e_4 \otimes e_4$ maps to:

$$\begin{aligned}\phi(f) &= \langle \Psi, f \otimes f \otimes f\rangle \\ &= c_{144}\langle e_1 \otimes e_4 \otimes e_4, f \otimes f \otimes f\rangle + c_{444}\langle e_4 \otimes e_4 \otimes e_4, f \otimes f \otimes f\rangle \\ &= c_{144}\langle e_1, f\rangle\langle e_4, f\rangle^2 + c_{444}\langle e_4, f\rangle^3 \ .\end{aligned}$$

The mapping can, of course, be restricted to the symmetric or anti-symmetric subspace of each n-particle Hilbert space $\mathscr{H}_{m,s}^{\otimes_n}$, as appropriate.

Whilst the elements of the n-particle subspace $\mathscr{H}_{m,s}^{\otimes_n} \subset \mathscr{F}_{m,s}$ map to n-degree monomials, elements of the Fock space $\mathscr{F}_{m,s}$ which contain components in more than one n-particle summand, map to polynomials. A 'finite particle vector' is an element of Fock space $\psi = \oplus_{n=0}^{\infty}\psi_n$ in which all but a finite number of the ψ_n are zero. The value of the corresponding polynomial on an arbitrary element f of the 1-particle subspace is given by:

$$\phi(f) = \sum_{n=0}^{\infty}\langle \psi_n, f^n\rangle \ ,$$

where f^n is the n-fold tensor product of f with itself,

$$f^n = f \otimes \cdots \otimes f \quad \text{(n times)} \ .$$

The alternative definition of the quantum field state space will occasionally provide an interesting perspective, but for the most part below we will focus on the Fock space approach.

Having defined the relevant Fock space, the next step is define creation and annihilation operators, and from these to construct field operators. Suppose one has a bosonic Fock space. For each $n-1$-particle Hilbert space $\mathscr{H}_{m,s}^{\odot_{n-1}}$, the creation of a particle with a state $f \in \mathscr{H}_{m,s}$ corresponds to the operator $a_n^*(f) : \mathscr{H}_{m,s}^{\odot_{n-1}} \to \mathscr{H}_{m,s}^{\odot_n}$ defined by

$$a_n^*(f)(f_1 \odot \cdots \odot f_{n-1}) = S_n^+(\sqrt{n+1}\, f \otimes f_1 \otimes \cdots \otimes f_{n-1}) \ ,$$

where S_n^+ is the projection operator onto the symmetric n-particle subspace, so that

$$S_n^+(\sqrt{n+1}\, f \otimes f_1 \otimes \cdots \otimes f_{n-1}) = f \odot f_1 \odot \cdots \odot f_{n-1} \ .$$

For each n-particle Hilbert space $\mathscr{H}_{m,s}^{\odot_n}$ the annihilation of a particle with a state f corresponds to the adjoint operator $a_n(f) : \mathscr{H}_{m,s}^{\odot_n} \to \mathscr{H}_{m,s}^{\odot_{n-1}}$, defined by

$$a_n(f)(f_1 \odot \cdots \odot f_n) = S_n^+(\sqrt{n}\, \langle f, f_1 \rangle f_2 \otimes f_n) \ .$$

These creation and annihilation operators are operator-valued distributions in the sense that they assign operators to functions f.

At various points in this paper it will be convenient to use the physicists' 'bra-ket' notation in association with the application of creation and annihilation operators. In this respect, for idealised systems with only one possible mode of excitation, an n-particle state is typically denoted as $|n\rangle$, with

$$|n+1\rangle = \frac{1}{\sqrt{n+1}} a^* |n\rangle \ .$$

One particular implication of this is that:

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^*)^n |\Omega\rangle \ ,$$

where Ω is the zero-particle vacuum vector we will meet in the next sub-section.

The annihilation and creation operators on the bosonic Fock space satisfy canonical commutation relations (CCRs):

$$\begin{aligned} a(f)a(g) - a(g)a(f) &= 0 \\ a^*(f)a^*(g) - a^*(g)a^*(f) &= 0 \\ a(f)a^*(g) - a^*(g)a(f) &= \langle f, g \rangle I \ . \end{aligned}$$

Note that the creation operators and annihilation operators on fermionic Fock space satisfy the canonical anti-commutation relations (CARs):

$$\begin{aligned} a(f)a(g) + a(g)a(f) &= 0 \\ a^*(f)a^*(g) + a^*(g)a^*(f) &= 0 \\ a(f)a^*(g) + a^*(g)a(f) &= \langle f, g \rangle I \ . \end{aligned}$$

This distinction will become crucial at a later stage when we turn to the subject of coherent states.

Given the creation and annihilation operators, one can try to define field operators at each point x of Minkowski space-time by expressions such as

$$A(x) = \sum_{\alpha=1}^{\infty} [f_\alpha(x) a(f_\alpha) + \bar{f}_\alpha(x) a^*(\bar{f}_\alpha)] \ ,$$

where the $\{f_1, f_2, ...\}$ provide an orthonormal basis of the single-particle Hilbert space. However, the second part of this series diverges, and, in general, quantum

field operators cannot be well-defined at individual points of Minkowski space-time. Instead, one must treat the field operators as operator-valued distributions $A(f)$ by 'smearing' them over functions f from a test-function space as follows,

$$\begin{aligned} A(f) &= \int_{\mathcal{M}} f(x)A(x)d^4x \\ &= \underset{n\to\infty}{\text{w-lim}} \int_{\mathcal{M}} f(x) \sum_{\alpha=1}^{n} [f_\alpha(x)a(f_\alpha) + \bar{f}_\alpha(x)a^*(\bar{f}_\alpha)]d^4x \ , \end{aligned}$$

where both of the terms exist as weak limits[5] on a dense set in Fock space as $n \to \infty$, (Prugovecki 1995, p155).

The test function space for the quantized radiation field is a complexified space of electromagnetic vector potentials; for quantization in the Coulomb gauge it is a set of divergence-free functions valued in $\mathbb{C}^3$. The test-functions $f(x)$ are required to be infinitely differentiable, and both they, and all their partial derivatives, must tend towards zero faster than a polynomial function x^{-n}, for any $n \in \mathbb{N}$, as $|x| \to \infty$.

Despite the mathematical necessity for smearing the field operators, it is conventional in the physics literature to maintain the use of notation in which field operators are assigned to points of space-time, and this paper will uphold that tradition. Use of such notation should be taken as a shorthand to indicate that the field operator $A(x)$ has been smeared with a test function that either has support in a small open neighbourhood of x, or which decays very rapidly inside that neighbourhood. For example, one might multiply some polynomial function by an exponential decay factor $e^{-a||x||^2}$.

[5] "The weak limit of a sequence of operators $A_1, A_2, ...$ in a Hilbert space is determined by the weak limits of the corresponding sequences $A_1 f, A_2 f, ...$ for all vectors f in their domains. In turn, a vector h is the weak limit of a sequence of vectors $h_1, h_2, ...$ in a Hilbert space if $\langle g|h\rangle$ is the limit of $\langle g|h_1\rangle, \langle g|h_2\rangle, ...$ for *any* vector g in that space," (Prugovecki 1995, p155, footnote 2).

3.2 The quantum vacuum

In both fermionic and bosonic Fock spaces the zero-particle subspace is $\mathscr{H}^0 = \mathbb{C}^1$, the so-called vacuum sector. This subspace contains a distinguished non-zero vector $1 \in \mathbb{C}^1$, called the vacuum vector. The vacuum vector is denoted as $\mathbf{0}$ or $|\mathbf{0}\rangle$ in some of the quantum field theory literature, despite the fact that it is not the zero vector in Fock space. To distinguish it from the zero vector, it is more usefully denoted as Ω.

The Fock space representations of the canonical commutation relations are defined by the requirement that there is a unique vector Ω which is such that $a(f)\Omega = 0$ for all f. The Fock space vacuum vector is cyclic with respect to the algebra generated by the representation of the commutation relations.

The vacuum vector Ω is such that

1. Ω is the ground state of the quantum field. It is the minimum energy eigenstate of the Hamiltonian.

2. Ω is a state with zero particles.

3. Ω is a state which is invariant under the unitary action of the Poincaré group, hence it is a state which is shared by all *inertial* observers.

On each bosonic and fermionic Fock space there is an operator called the number operator,

$$N = 0 \oplus 1I \oplus 2I \oplus 3I \oplus \cdots \ .$$

The eigenstates of this operator are often considered to represent the states of the second-quantized theory in which there are a definite number of particles. The utility of Fock space is that it enables one to represent situations where there is (i) a variable number of particles, or (ii) an indefinite number of particles.

The vacuum state of the free electromagnetic field is not an eigenstate of any of the local field operators representing the electric field $\widehat{\mathbf{E}}(x)$ or the magnetic field $\widehat{\mathbf{B}}(x)$ at points x of space-time. An eigenstate Ψ of a self-adjoint operator A is defined to be 'dispersion-free', where 'dispersion' is used as a synonym for 'variance'.

The variance of a quantity is defined to be the mean value of $(A - \langle A\rangle)^2$, where $\langle A\rangle$ denotes the mean value. (In the special case of a quantity with a mean value of zero, the variance becomes the mean value of A^2). In quantum theory, the variance in the state Ψ is the expectation value:

$$\langle \Psi, (\widehat{A} - \langle \widehat{A}\rangle)^2 \Psi\rangle \ .$$

In the vacuum state, the mean values of the electromagnetic field operators are zero:

$$\langle \Omega, \widehat{\mathbf{E}}(x)\Omega\rangle = \langle \Omega, \widehat{\mathbf{B}}(x)\Omega\rangle = 0 \ .$$

Hence, the variance of the electric and magnetic field is given by expectation values of the squared fields, and these are non-zero (Rugh and Zinkernagel, 2002):

$$\langle \Omega, \widehat{\mathbf{E}}^2(x)\Omega\rangle \neq 0$$
$$\langle \Omega, \widehat{\mathbf{B}}^2(x)\Omega\rangle \neq 0 \ .$$

Hence, the local field operators of the free electromagnetic field are not dispersion-free in the vacuum state. Equivalently, the relativistic vacuum state of the electromagnetic field is not an eigenstate of the local field operators. In this sense, the vacuum state of the free electromagnetic field contains 'fluctuations'.

In fact, none of the eigenstates of the photon number operator are also eigenstates of the local field operators. The photon number operator doesn't commute with the field operators. Hence, the electric and magnetic fields are subject to fluctuations whenever the state contains a definite number of photons.

Whilst the vacuum state is the state of *minimum* energy, it is not *necessarily* a state of zero energy. This purported non-zero energy of the vacuum state is referred to as the 'zero-point energy'.

Given that the Hamiltonian density of the electromagnetic field is $H(x) = \frac{1}{2}(\mathbf{E}^2(x) + c^2\mathbf{B}^2(x))$, the zero point energy of the electromagnetic field per unit volume is (Rugh and Zinkernagel 2002; Aitchison 1985, p347):

$$\langle \Omega, \widehat{H}\Omega\rangle = \langle \Omega, \int \tfrac{1}{2}(\widehat{\mathbf{E}}^2(x) + c^2\widehat{\mathbf{B}}^2(x))d^3x\ \Omega\rangle = \sum_{\epsilon}\sum_{\mathbf{k}} \tfrac{1}{2}\hbar\omega_{\mathbf{k},\epsilon} \ .$$

To obtain a finite value, the electromagnetic field has been confined to a cubical box of finite volume $V = L^3$. The sum on the right is over the two independent polarization degrees of freedom $\epsilon = 1, 2$, and the *normal* modes $\mathbf{k}$ up to an 'ultraviolet cut-off' $||\mathbf{k}||_{max}$. i.e.,

$$\mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^3, \qquad ||\mathbf{k}|| < ||\mathbf{k}||_{max} \ .$$

However, it is conventional for the Hamiltonian to be normally ordered in quantum field theory calculations, (i.e., the order of annihilation operators and creation operators are swapped wherever they appear as products with creation operators to the right of the annihilation operators). Using the : : notation to indicate normal ordering, this results in the disappearance of the zero-point energy,

$$\langle \Omega, : \widehat{H} : \Omega\rangle = 0 \ .$$

Note that the vacuum state defined here is the vacuum state of the free electromagnetic field. It is *not* the vacuum state of the interacting electromagnetic and electron-positron fields. We will discuss the interacting field vacuum in a later section.

3.3 The quantized radiation field

The basic technique for quantizing a field is to express the field as an inverse Fourier transform, and then to substitute creation and annihilation operators in the place of the Fourier coefficients which specify the amplitude of each mode. These field operators then act on the Fock space constructed from the relevant one-particle Hilbert space; i.e., the irreducible unitary representation of the space-time symmetry group corresponding to the excited states ('quanta') of the field.

In the case of the electromagnetic field, this process is typically applied first to the electromagnetic potential $A_\mu(x)$ rather than the electric and magnetic fields $\mathbf{E}(x), \mathbf{B}(x)$. However, because an entire gauge equivalence class of potentials correspond to a single pair $\mathbf{E}(x), \mathbf{B}(x)$, constraints must be applied at some stage of the process, and there are different ways of doing this.

In the Gupta-Bleuler method, the Lorenz gauge condition is applied after a provisional quantum state space has been obtained. The approach begins by taking the space of all real vector potentials $A_\mu(x)$ which solve the wave-equation, and complexifying it into a space of cross-sections $f_\mu(x)$ of the complexified cotangent bundle. The Fourier transform $\tilde{f}_\mu(k)$ of each element in this space is concentrated on the light cone in energy-momentum space.

This space is then equipped with an inner product. However, the inner product is indefinite, hence this state space is not a Hilbert space proper. It is, instead, a Krein space $\mathscr{H}_K$, or 'pseudo-Hilbert' space. It has a direct sum decomposition (Prugovecki 1995, p237):

$$\mathscr{H}_K = \mathscr{H}^- \oplus \mathscr{H}^+ \,,$$

where

$$\mathscr{H}^- = \{f : f_a = 0, a = 1, 2, 3\}, \qquad \mathscr{H}^+ = \{f : f_0 = 0\}.$$

The inner product (and norm) is positive definite on $\mathscr{H}^+$, and negative definite on $\mathscr{H}^-$.

This Krein space is a space of provisional one-particle states. The one-particle states are photon states, and three types of photons are identifiable in this state-space. If $f \in \mathscr{H}^-$, then $a^*(f)$ creates a so-called 'scalar photon'. The positive-definite subspace of the Krein space can be decomposed into a transverse subspace, $\mathscr{H}^\perp = \{f : f_3 = 0\}$, and a longitudinal subspace $\mathscr{H}^\parallel = \{f : f_1 = f_2 = 0\}$. If $f \in \mathscr{H}^\perp$, then $a^*(f)$ creates a so-called 'transverse photon', and if $f \in \mathscr{H}^\parallel$, then $a^*(f)$ creates a so-called 'longitudinal photon'. Scalar photons and longitudinal photons, however, are considered to be unphysical states of the free field.

Having obtained this Krein space, it can be treated as the one-particle space of the *unconstrained* theory. To obtain a *physical* state space, one imposes the Lorenz gauge condition. In terms of real 4-vector potentials, this is the stipulation that:

$$\partial^\mu A_\mu = -\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A} = 0 \,.$$

The space of such potentials is Lorentz invariant. i.e., a Lorentz transformation maps one Lorenz gauge 4-potential into another Lorenz gauge 4-potential.

To apply this gauge constraint, a Krein-Fock space is constructed from the one-particle Krein space $\mathscr{H}_K$, and one can define operators which create $\widehat{a}^*_\mu(\mathbf{k},\epsilon)$ and annihilate $\widehat{a}_\mu(\mathbf{k},\epsilon)$ one-particle states indexed by momentum $\mathbf{k}$ and polarization ϵ. Field operators $\widehat{A}_\mu$ can thence be constructed from the creation and annihilation operators:

$$\begin{aligned}\widehat{A}_\mu(x) &= \frac{1}{(2\pi)^3}\sum_{i=0}^{3}\int_{\mathscr{V}_0^+}\varepsilon(\mathbf{k},\epsilon_i)[e^{i(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t)}\widehat{a}_\mu(\mathbf{k},\epsilon_i) + \ e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t)}\widehat{a}^*_\mu(\mathbf{k},\epsilon_i)]\ \frac{d^3\mathbf{k}}{2\omega(\mathbf{k})} \\ &= \widehat{A}^+_\mu + \widehat{A}^-_\mu\ ,\end{aligned}$$

where $\{\varepsilon(\mathbf{0},\epsilon_i) : i = 0,1,2,3\}$ is an orthonormal basis for $\mathbb{C}^4$.

The Lorenz gauge constraint is the requirement that physical states $|f\rangle$ satisfy the equation:

$$\left(-\frac{\partial\widehat{A}_0}{\partial t} + \nabla\cdot\widehat{\mathbf{A}}\right)^+|f\rangle = \mathbf{0}\ ,$$

where the + superscript indicates that the positive-frequency part of the operator is taken. This part of the operator is constructed from an inverse Fourier transform of photon annihilation operators, hence it is the component of the field capable of mapping states f which contain particles into the zero vector.

If $\widehat{A}^+_\mu$ is substituted into this equation it follows that:

$$(\widehat{a}_3(\mathbf{k},\epsilon_i) - \widehat{a}_0(\mathbf{k},\epsilon_i))|f\rangle = \mathbf{0}\ .$$

This entails that a state which contains a longitudinal photon of momentum $\mathbf{k}$ must also contain a scalar photon of the same momentum. This eliminates the states of negative norm. The states of zero norm are then eliminated by defining an equivalence relationship, so that states which only differ in their longitudinal and scalar photon counts are physically equivalent. The set of equivalence classes duly possesses a positive-definite inner product.

There is a variation on this method of applying the Lorenz gauge constraint. Here we extract the salient points from the exposition of Prugovecki (1995, pp237-240). Once the one-particle Krein space is obtained, an immediate restriction to a subspace satisfying the Lorenz gauge is imposed. Recall that the Lorenz gauge condition on the $f_\mu(x)$ in the configuration representation is defined by the expression:

$$\partial^\mu f_\mu(x) = 0\ .$$

On the Fourier-transformed Krein space, $\tilde{\mathscr{H}}_K$, it is defined by the condition:

$$k^\mu\tilde{f}_\mu(k) = 0\ .$$

One can choose a basis $\{e^a(k) : a = 0,1,2,3\}$ for the fibre of the contangent bundle over each point k of the forward light cone which is such that $k^\mu e^1_\mu(k) =$

$k^\mu e^2_\mu(k) = 0$, and such that $k = k^0(e^0(k) + e^3(k))$. In this basis, $\tilde{f}(k) = \tilde{f}_a(k)e^a(k)$, and the Lorenz gauge condition is equivalent to the restriction:

$$\tilde{\mathscr{H}}_{Lorenz} = \{\tilde{f}(k) \in \tilde{\mathscr{H}}_K : \tilde{f}_0(k) = \tilde{f}_3(k)\} \ .$$

There is a further subspace $\tilde{\mathscr{H}}_0 \subset \tilde{\mathscr{H}}_{Lorenz}$ such that:

$$\tilde{\mathscr{H}}_0 = \{\tilde{f}(k) \in \tilde{\mathscr{H}}_{Lorenz} : \tilde{f}_1(k) = \tilde{f}_2(k) = 0\} \ .$$

These are the complexified analogues of the real vector potentials which correspond to a vanishing electromagnetic field $F_{\mu\nu} = 0$. This is the subspace of elements with zero norm, the 'null' subspace.

The direct sum decomposition of the Krein space $\tilde{\mathscr{H}}_K = \tilde{\mathscr{H}}^- \oplus \tilde{\mathscr{H}}^+$ reduces to the following direct sum decomposition of the subspace satisfying the Lorenz gauge condition:

$$\tilde{\mathscr{H}}_{Lorenz} = \tilde{\mathscr{H}}_0 \oplus \tilde{\mathscr{H}}_\perp \ .$$

The first summand is the null subspace, and the second is the 'transverse' subspace, the subspace of $\tilde{\mathscr{H}}^+$ satisfying the Lorenz gauge condition:

$$\tilde{\mathscr{H}}_\perp = \{\tilde{f}(k) \in \tilde{\mathscr{H}}^+ : \tilde{f}_0(k) = \tilde{f}_3(k) = 0\} \ .$$

By taking the quotient,

$$\tilde{\mathscr{H}}_{phys} = \tilde{\mathscr{H}}_{Lorenz}/\tilde{\mathscr{H}}_0 \ ,$$

one obtains the physical one-particle space, equipped with a positive-definite inner product. The members of each equivalence class differ only by virtue of their scalar $\tilde{f}_0(k)$ and longitudinal components $\tilde{f}_3(k)$. Each equivalence class can be mapped to a pair of transverse components $(\tilde{f}_1(k), \tilde{f}_2(k))$.

The Gupta-Bleuler approach to quantization of the free electromagnetic field is Lorentz-invariant. An alternative approach to obtaining the physical space of transverse photons is to select the Coulomb gauge, (or 'radiation gauge'), defined by the following joint condition:

$$A = (0, \mathbf{A}) \qquad \nabla \cdot \mathbf{A} = 0 \ .$$

The first condition alone is referred to as 'temporal gauge'. This joint condition is *not* Lorentz invariant, but it is convenient for quantizing the free electromagnetic field.

By definition, a vector field $\mathbf{A}$ can be decomposed into the sum $\mathbf{A} = \mathbf{A}^\perp + \mathbf{A}^\parallel$ of a transverse component and a longitudinal component by the conditions,

$$\nabla \cdot \mathbf{A}^\perp = 0 \ , \qquad \nabla \times \mathbf{A}^\parallel = 0 \ .$$

Hence, in the Coulomb gauge, the 3-vector potential is purely transverse, $\mathbf{A} = \mathbf{A}^\perp$.

The electric field is related to the 4-vector potential as follows:

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \ .$$

The transverse and longitudinal components of the electric field are as follows (Honegger and Rieckers, p1988):

$$\mathbf{E}^{\perp} = -\frac{\partial \mathbf{A}^{\perp}}{\partial t}$$
$$\mathbf{E}^{\parallel} = -\nabla A_0 - \frac{\partial \mathbf{A}^{\parallel}}{\partial t}$$

The temporal gauge condition, $A_0 = 0$, entails that the electric field reduces to:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \ .$$

Taking the divergence of this yields:

$$\nabla \cdot \mathbf{E} = -\frac{\partial \nabla \cdot \mathbf{A}}{\partial t} \ .$$

Given that $\nabla \cdot \mathbf{A} = 0$ in the Coulomb gauge, it follows that:

$$\nabla \cdot \mathbf{E} = 0 \ .$$

This is Gauss's law in the case of a free field. Or, to put it another way, the electric field vector of the free field is transverse, $\mathbf{E} = \mathbf{E}^{\perp}$.

In the Coulomb gauge, the real 3-dimensional polarization vector ε is constrained to be orthogonal to the propagation vector $\mathbf{k}$, hence one obtains a Fourier decomposition of the vector potential:

$$\mathbf{A}(x) = \frac{1}{(2\pi)^3} \sum_{i=1}^{2} \int_{\mathscr{V}_0^+} \varepsilon(\mathbf{k}, \epsilon_i)[e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} a(\mathbf{k}, \epsilon_i) + \ e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} a^*(\mathbf{k}, \epsilon_i)] \ \frac{d^3\mathbf{k}}{2\omega(\mathbf{k})}.$$

Note that $\varepsilon(\mathbf{k}, \epsilon_i)$ denotes the real 3-dimensional polarization vector associated with the mode $(\mathbf{k}, \epsilon_i)$, defined by combination of wave-vector $\mathbf{k}$ and complex 2-dimensional polarization basis vector ϵ_i.

Substituting creation and annihilation operators in place of the Fourier coefficients one obtains the expression for the field operator:

$$\begin{aligned}\widehat{\mathbf{A}}(x) &= \frac{1}{(2\pi)^3} \sum_{i=1}^{2} \int_{\mathscr{V}_0^+} \varepsilon(\mathbf{k}, \epsilon_i)[e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} \widehat{a}(\mathbf{k}, \epsilon_i) + \ e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} \widehat{a}^*(\mathbf{k}, \epsilon_i)] \ \frac{d^3\mathbf{k}}{2\omega(\mathbf{k})} \\ &= \frac{1}{(2\pi)^3} \sum_{i=1}^{2} \int_{\mathscr{V}_0^+} \widehat{A}_{\mathbf{k},\epsilon_i} \ \frac{d^3\mathbf{k}}{2\omega(\mathbf{k})} \ .\end{aligned}$$

4 Interactions between light and matter

4.1 The coupled electromagnetic field

The quantized radiation field is equivalent to a quantization of the free electromagnetic field. In contrast, the subject of this section is the electromagnetic field interacting with a charge-current density $(\rho, \mathbf{j})$.

In terms of the 4-vector potential, the Maxwell equations become the following:

$$-\nabla^2 A_0 - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{\rho}{\epsilon_0}$$
$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A} - \frac{\partial A_0}{\partial t}) = \frac{\mathbf{j}}{\epsilon_0} \ ,$$

where ϵ_0 is the permittivity of free space. From the first equation it follows that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \ .$$

In the presence of a non-zero charge density, the fact that $\nabla \cdot \mathbf{E} \neq 0$ entails that the electric field is not purely transverse. In fact, the Gauss law becomes an equation for the longitudinal component $\mathbf{E}^{\parallel}$ of the electric field because:

$$\nabla \cdot \mathbf{E} = \nabla \cdot (\mathbf{E}^{\perp} + \mathbf{E}^{\parallel}) = \nabla \cdot \mathbf{E}^{\parallel} = \frac{\rho}{\epsilon_0} \ .$$

It follows that

$$\mathbf{E}^{\parallel}(\mathbf{x}, t) = -\frac{1}{4\pi\epsilon_0} \nabla \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y}, t)}{|\mathbf{x} - \mathbf{y}|} \, d^3\mathbf{y} \ .$$

Hence, the longitudinal component $\mathbf{E}^{\parallel}$ of the electric field at a point in space is determined by an integral of the charge density ρ taken over all of space. Even if the charge density is confined to a bounded open subset of space, it will determine the longitudinal electric field throughout all of the surrounding space. When the theory is quantized, the distribution of expectation values of the longitudinal electric field and charge density will possess the same relationship:

$$\langle \widehat{\mathbf{E}}^{\parallel}(\mathbf{x}, t) \rangle = -\frac{1}{4\pi\epsilon_0} \nabla \int_{\mathbb{R}^3} \frac{\langle \widehat{\rho}(\mathbf{y}, t) \rangle}{|\mathbf{x} - \mathbf{y}|} \, d^3\mathbf{y} \ .$$

The longitudinal component of the electric field represents the Coulomb field. It is typically referred to as the *electrostatic* Coulomb field, but note that $\mathbf{E}^{\parallel}$ is permitted to evolve in accordance with:

$$\frac{\partial \mathbf{E}^{\parallel}}{\partial t} = -\frac{1}{\epsilon_0} \mathbf{j}^{\parallel} \ .$$

There is no violation of relativistic causality; the electric field as a whole satisfies the following wave-equation (Honegger and Rieckers, p2068):

$$\Box \mathbf{E} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} - \frac{1}{\epsilon_0} \nabla \rho \ ,$$

where μ_0 is the permeability of free space, and

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle \, .$$

(Whilst we have tacitly been using 'relativistic units', in which $c = 1$, the speed of light has been re-introduced in this expression to make the relativistic causality explicit).

The gradient of the charge density is longitudinal, and the wave equation reduces to separate wave equations for the transverse and longitudinal components of the electric field, (ibid, p2069):

$$\Box \mathbf{E}^{\perp} = -\mu_0 \frac{\partial \mathbf{j}^{\perp}}{\partial t}$$
$$\Box \mathbf{E}^{\parallel} = -\mu_0 \frac{\partial \mathbf{j}^{\parallel}}{\partial t} - \frac{1}{\epsilon_0}\nabla\rho \, .$$

Further sources of potential confusion can arise when specific gauges are chosen, so let's briefly try to clarify these matters.

Consider again the two equations at the beginning of the section for the 4-vector potential. Let's apply the Lorenz gauge:

$$-\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A} = 0 \, .$$

Reverting to 'natural' units in which both $c = 1$ and $\epsilon_0 = 1$, the equations for the 4-vector potential reduce to the following:

$$-\frac{\partial^2 A_0}{\partial t^2} + \nabla^2 A_0 = \rho$$
$$-\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = \mathbf{j} \, ,$$

Hence, in the Lorenz gauge, both the scalar potential A_0 and the vector potential $\mathbf{A}$ satisfy a wave-equation with source. Thus, in this gauge, changes in the charge distribution propagate through the scalar potential at the speed of light, and changes in the current propagate through the vector potential at the speed of light.

In the Lorenz gauge, then, the potentials behave as a physical field would. This is deceptive, however. If, instead of taking the Lorenz gauge, we impose one part of the Coulomb gauge condition, $\nabla \cdot \mathbf{A} = 0$, then the first equation becomes Poisson's equation:

$$\nabla^2 A_0 = -\rho \, ,$$

and the second equation becomes

$$-\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = \mathbf{j} - \nabla \frac{\partial A_0}{\partial t} \, .$$

The condition that $\nabla \cdot \mathbf{A} = 0$ also entails that $\mathbf{j}^{\perp} = \mathbf{j} - \nabla \partial A_0 / \partial t$, (Grensing, p430), hence the second equation reduces to:

$$-\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = \mathbf{j}^{\perp} \ ,$$

a wave-equation with the transverse component of the current as source.

The appearance of the Poisson equation entails that under the gauge condition $\nabla \cdot \mathbf{A} = 0$, if the charge density ρ changes in time, then the scalar potential A_0 changes instantaneously to match it. However, the physical fields $(\mathbf{E}, \mathbf{B})$ do *not* change instantaneously. A change in charge density entails the existence of a non-zero current, and via the evolution of the vector potential $\mathbf{A}$, the physical fields $(\mathbf{E}, \mathbf{B})$ satisfy the strictures of relativistic causality.

4.2 Scattering, Feynman diagrams and virtual particles

We now turn to the representation of interactions in quantum field theory. Although the states in Fock space are the states of a free system, by defining a scattering operator S one can calculate the transition probabilities $\langle \psi_{out} \mid S\psi_{in} \rangle$ between the asymptotically free incoming states ψ_{in} and outgoing states ψ_{out} in a collision or decay process, special types of interaction in which the coupling is transient and spatially localised.

The scattering operator is defined in terms of an interaction Hamiltonian density operator $\widehat{H}_I(x)$. The total Hamiltonian of an interacting system can be broken up into the free Hamiltonian operator $\widehat{H}_0(x)$, and the interaction Hamiltonian operator $\widehat{H}_I(x)$.

The scattering operator S can be expressed in a Dyson perturbation expansion as

$$\begin{aligned} S &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^4} d^4x_1 \cdots \int_{\mathbb{R}^4} d^4x_n T[\widehat{H}_I(x_1) \cdots \widehat{H}_I(x_n)] \\ &= \sum_{n=0}^{\infty} S_n \ . \end{aligned}$$

$\widehat{H}_I(x)$ is the interaction Hamiltonian density operator, and $T[\widehat{H}_I(x_1) \cdots \widehat{H}_I(x_n)]$ is a time-ordered permutation[6] in the sense that

$$T[\widehat{H}_I(x_1) \cdots \widehat{H}_I(x_n)] = \widehat{H}_I(x_{i_1}) \cdots \widehat{H}_I(x_{i_n}) \ ,$$

with $t(x_{i_1}) \geq t(x_{i_2}) \geq \cdots \geq t(x_{i_n})$. The interaction Hamiltonian density operator is expressed in terms of the field operators relevant to the interaction in question. In the case of quantum electrodynamics (QED), for example, one has field operators $\widehat{\Psi}(x)$ for the second-quantized Dirac field, field operators

[6] When the Hamiltonian includes fermion field operators, one must introduce the sign of the permutation into the expression for the time-ordering.

$\widehat{\overline{\Psi}}(x)$ for the second-quantized conjugate Dirac field, and field operators for the second-quantized electromagnetic field $\widehat{A}_\mu(x)$, (Teller 1995, p124-125).

The creation operators, annihilation operators and field operators for an electromagnetic field are defined upon a photonic Fock space:

$$\mathscr{F}_\gamma = \mathscr{F}(\mathscr{H}_{0,1})$$

Similarly, the creation operators, annihilation operators and field operators of a Dirac field are defined upon the electron-positron Fock space $\mathscr{F}_{e,e^+}$. This can be obtained either by taking the Fock space of the direct sum of the electron and positron single-particle spaces,

$$\mathscr{F}_{e,e^+} = \mathscr{F}(\mathscr{H}^+_{m_e,1/2} \oplus \mathscr{H}^-_{m_{e^+},1/2}) \ ,$$

or by taking the direct sum of the Fock spaces for the electron and positron,

$$\mathscr{F}_{e,e^+} = \mathscr{F}(\mathscr{H}^+_{m_e,1/2}) \oplus \mathscr{F}(\mathscr{H}^-_{m_{e^+},1/2}) \ .$$

For quantum electrodynamics, one takes the tensor product $\mathscr{F}_\gamma \otimes \mathscr{F}_{e,e^+}$ of these two Fock spaces. The operators of the individual fields are extended to the entire tensor product space in a trivial manner. For example, the Dirac field operator $\widehat{\Psi}(x)$ defined upon $\mathscr{F}_{e,e^+}$ is extended to the operator $I \otimes \widehat{\Psi}(x)$ defined upon $\mathscr{F}_\gamma \otimes \mathscr{F}_{e,e^+}$. This fact can be conveniently suppressed from most of the notation, but is important to bear in mind given that the interaction Hamiltonian density operator is a product of field operators from $\mathscr{F}_{e,e^+}$ and field operators from $\mathscr{F}_\gamma$. The interaction Hamiltonian density operator is only defined upon the tensor product Fock space $\mathscr{F}_\gamma \otimes \mathscr{F}_{e,e^+}$. It is obtained by substituting field operators into the expression for the classical interaction Hamiltonian density, and subjecting them to 'normal ordering', (see Teller 1995, p127-129).

In the case of a Dirac field interacting with an electromagnetic field, the interaction term in the Lagrangian density provides the interaction Hamiltonian density. Thus, in the case of quantum electrodynamics, and prior to normal ordering, one obtains the following interaction Hamiltonian density operator:

$$\widehat{H}_I(x) = q\widehat{\overline{\Psi}}(x)\gamma^\mu\widehat{\Psi}(x)\widehat{A}_\mu(x) \ ,$$

where q is the charge of the electron, and γ^μ is a Dirac gamma matrix.

Even with the interaction Hamiltonian density operator for quantum electrodynamics inserted into the expression for the scattering operator, it is still impossible to *rigorously* calculate the transition amplitudes between the asymptotically free incoming states ψ_{in} and the asymptotically free outgoing states ψ_{out} of electrons, positrons and photons, because the following perturbation series is divergent:

$$\langle \psi_{out} \mid S\psi_{in} \rangle = \langle \psi_{out} \mid I\psi_{in} \rangle + \langle \psi_{out} \mid S_1\psi_{in} \rangle + \langle \psi_{out} \mid S_2\psi_{in} \rangle + \cdots \ .$$

Physicists respond to this difficulty by assuming, rather than proving, that this is a divergent asymptotic series. The significance of an asymptotic series is that

the first few terms can approximate the 'true value' one is trying to calculate, even if the series as a whole diverges. Hence, assuming the series is asymptotic, physicists calculate only, say, the first and second-order terms.

Under these assumptions, quantum field theory can be used to calculate the transition probabilities $\langle \psi_{out} \mid S\psi_{in} \rangle$ between matter field states and gauge field states of a specific energy and momentum, as well as a specific particle number. As an illustration, let's consider Compton scattering, the scattering of a single electron with a single photon (Teller 1995, p132-133).

An incoming state in which the electron has 4-momentum[7] $(\omega(\mathbf{p}_i), \mathbf{p}_i)$ and spin r_i, and the photon has 4-momentum[8] $(\omega(\mathbf{k}_i), \mathbf{k}_i)$ and polarization λ_i, is represented as $\psi_{in} = \widehat{a}^*_{\lambda_i}(\mathbf{k}_i)\widehat{b}^*_{r_i}(\mathbf{p}_i)\,\Omega$, where Ω is the vacuum vector, $\widehat{b}^*_{r_i}(\mathbf{p}_i)$ is the creation operator for an electron of 4-momentum $(\omega(\mathbf{p}_i), \mathbf{p}_i)$ and spin r_i, whilst $\widehat{a}^*_{\lambda_i}(\mathbf{k}_i)$ is the creation operator for a photon of 4-momentum $(\omega(\mathbf{k}_i), \mathbf{k}_i)$ and polarization λ_i.[9] An outgoing state in which the electron has 4-momentum $(\omega(\mathbf{p}_o), \mathbf{p}_o)$ and spin r_o, whilst the photon has 4-momentum $(\omega(\mathbf{k}_o), \mathbf{k}_o)$ and polarization λ_o, is represented as $\psi_{out} = \widehat{a}^*_{\lambda_o}(\mathbf{k}_o)\widehat{a}^*_{r_o}(\mathbf{p}_o)\,\Omega$.

In terms of the tensor product Fock space $\mathscr{F}_\gamma \otimes \mathscr{F}_{e,e^+}$, these incoming and outgoing states can be represented as

$$\begin{aligned}\psi_{in} &= \widehat{a}^*_{\lambda_i}(\mathbf{k}_i) \otimes \widehat{b}^*_{r_i}(\mathbf{p}_i)(\Omega_\gamma \otimes \Omega_{e,e^+}) \\ &= (\widehat{a}^*_{\lambda_i}(\mathbf{k}_i)\,\Omega_\gamma) \otimes (\widehat{b}^*_{r_i}(\mathbf{p}_i)\,\Omega_{e,e^+})\ ,\end{aligned}$$

and

$$\begin{aligned}\psi_{out} &= \widehat{a}^*_{\lambda_o}(\mathbf{k}_o) \otimes \widehat{b}^*_{r_o}(\mathbf{p}_o)(\Omega_\gamma \otimes \Omega_{e,e^+}) \\ &= (\widehat{a}^*_{\lambda_o}(\mathbf{k}_o)\,\Omega_\gamma) \otimes (\widehat{b}^*_{r_o}(\mathbf{p}_o)\,\Omega_{e,e^+})\ ,\end{aligned}$$

where Ω_γ is the vacuum vector of the photonic Fock space, and Ω_{e,e^+} is the vacuum vector of the electron-positron Fock space.

The transition probability can be written as

$$\begin{aligned}\langle \psi_{out} \mid S\psi_{in} \rangle &= \langle \widehat{a}^*_{\lambda_o}(\mathbf{k}_o)\widehat{b}^*_{r_o}(\mathbf{p}_o)\Omega \mid S\,\widehat{a}^*_{\lambda_i}(\mathbf{k}_i)\widehat{b}^*_{r_i}(\mathbf{p}_i)\,\Omega \rangle \\ &= \langle \Omega \mid \widehat{b}_{r_o}(\mathbf{p}_o)\widehat{a}_{\lambda_o}(\mathbf{k}_o)\,S\,\widehat{a}^*_{\lambda_i}(\mathbf{k}_i)\widehat{b}^*_{r_i}(\mathbf{p}_i)\,\Omega \rangle\ .\end{aligned}$$

Let's consider the second-order term:

$$\langle \psi_{out} \mid S_2\psi_{in} \rangle = -\frac{1}{2}\int_{\mathbb{R}^4} d^4x_1 \int_{\mathbb{R}^4} d^4x_2 \langle \psi_{out} \mid T[\widehat{H}_I(x_1)\widehat{H}_I(x_2)]\psi_{in} \rangle\ .$$

The interaction Hamiltonian density operator is:

$$\begin{aligned}\widehat{H}_I(x) &= q\widehat{\overline{\Psi}}(x)\gamma^\mu\widehat{\Psi}(x)\widehat{A}_\mu(x) \\ &= q\left(\widehat{\overline{\Psi}}^+(x) + \widehat{\overline{\Psi}}^-(x)\right)\gamma^\mu\left(\widehat{\Psi}^+(x) + \widehat{\Psi}^-(x)\right)\left(\widehat{A}^+_\mu(x) + \widehat{A}^-_\mu(x)\right)\ .\end{aligned}$$

[7] With $\omega(\mathbf{p}_i) = +(m_e^2 + \|\mathbf{p}_i\|^2)^{1/2}$

[8] With $\omega(\mathbf{k}_i) = +\|\mathbf{k}_i\|$

[9] As a notational convenience, in the case of a photon we shall use 'k' rather than 'p' to denote the 4-momentum.

Recall that a product of field operators, such as that in the interaction Hamiltonian density operator, must be subjected to normal ordering, (denoted as $: \widehat{H}_I(x) :$), so that the order of the annihilation and creation operators is swapped in any terms where they occur as products with creation operators to the right of the annihilation operators.

The integrand of $\langle \psi_{out} \mid S_2 \psi_{in} \rangle$ is a sum of sixty-four terms, but only four of those are non-zero. Each of these four terms corresponds to a distinct process in Feynman diagram terms. Each can be evaluated by 'walking' the annihilation operators to the right of the expression. This technique follows from the commutation relations between annihilation operators, creation operators, and field operators. Letting i and j independently denote either a space-time index or a momentum-space index, these commutation relations are as follows (Teller 1995, p131):

$$[\widehat{a}_i, \widehat{a}_j^*]_{\pm} = c(i,j) \; .$$

This entails that

$$\widehat{a}_i \widehat{a}_j^* = c(i,j) \mp \widehat{a}_j^* \widehat{a}_i \; ,$$

hence

$$\langle \Omega \mid \cdots \widehat{a}_i \widehat{a}_j^* \cdots \Omega \rangle = c(i,j) \langle \Omega \mid \cdots \Omega \rangle \mp \langle \Omega \mid \cdots \widehat{a}_j^* \widehat{a}_i \cdots \Omega \rangle \; .$$

Repeated application of this procedure can be used to either 'walk' the annihilation operators to the right-hand side, where $\widehat{a}_i \Omega = 0$, or walk the creation operators to the left-hand side, where $\langle \Omega | \widehat{a}_j^* = 0$. This process leaves behind only a product of complex-valued functions ('c-numbers'), which can then be integrated over the space-time variables, which in the case of second-order Compton scattering are x_1 and x_2. Neglecting issues of time ordering, these commutators/c-numbers coincide with what are referred to in the physics literature as 'contractions':

$$\overbrace{\widehat{a}_i \; \widehat{a}_j^*} = [\widehat{a}_i, \widehat{a}_j^*]_{\pm} \; .$$

The technique extends to field operators, so that $\widehat{\Psi}^+(x_2)$, the positive-frequency component of the field, (and therefore the inverse Fourier transform of annihilation operators), is walked to the right until $\widehat{\Psi}^+(x_2)\Omega = \mathbf{0}$. Similarly, $\langle \Omega | \widehat{\Psi}^-(x_1) = \mathbf{0}$.

As an illustration, let's consider one of the four non-zero terms which contribute to the second-order amplitude for Compton scattering:

$$\left\langle \Omega \left| \widehat{b}_{r_o}(\mathbf{p}_o) \widehat{a}_{\lambda_o}(\mathbf{k}_o) \; T \left[\overbrace{\widehat{\Psi}}^{-}(x_1) \gamma^\mu \widehat{A}_\mu^-(x_1) \widehat{\Psi}^+(x_1) \overbrace{\widehat{\Psi}}^{-}(x_2) \gamma^\mu \widehat{A}_\mu^+(x_2) \widehat{\Psi}^+(x_2) \right] \widehat{a}_{\lambda_i}^*(\mathbf{k}_i) \widehat{b}_{r_i}^*(\mathbf{p}_i) \, \Omega \right\rangle \; .$$

This reduces to:

$$c(\mathbf{k}_o, x_1) c(\mathbf{p}_o, x_1) c(x_1, x_2) c(x_2, \mathbf{p}_i) c(x_2, \mathbf{k}_i) \langle \Omega \mid \Omega \rangle \; .$$

Given that $\langle\Omega|\Omega\rangle = 1$, this corresponds to a product of complex-valued functions. In total, the process described corresponds to the annihilation of a photon of momentum $\mathbf{k}_i$ and an electron of momentum $\mathbf{p}_i$ at a space-time point x_2, the creation of a virtual electron at x_2, and its annihilation at x_1, and the creation of a photon of momentum $\mathbf{k}_o$ and an electron of momentum $\mathbf{p}_o$ at a space-time point x_1.

To calculate the contribution this process makes to second-order Compton scattering, we then need to integrate this expression over the two arbitrary space-time points:

$$\int d^4x_1 \int d^4x_2 \; c(\mathbf{k}_o, x_1)c(\mathbf{p}_o, x_1)c(x_1, x_2)c(x_2, \mathbf{p}_i)c(x_2, \mathbf{k}_i) \; .$$

The Feynman diagrams beloved of textbooks on quantum field theory, offer a graphical mnemonic for this algorithmic procedure involved in calculating each term in such a perturbation series. However, each term in the perturbation series is itself a divergent integral, hence even the calculation of the first and second-order terms requires the use of so-called 'renormalization' to obtain finite results. Renormalization introduces factors into the integrands which enable the integrals to approach a finite value as the limits of the integrals are taken to infinity, (see Teller 1988).

Thus, as Berestetskii *et al* comment, "The lack of complete logical consistency in this theory [QED] is shown by the occurrence of divergent expressions when the mathematical formalism is directly applied, although there are quite well-defined ways of eliminating those divergencies. Nevertheless, such methods remain to a considerable extent, semiempirical rules, and our confidence in the correctness of the results is ultimately based only on their excellent agreement with experiment, not on the internal consistency or logical ordering of the fundamental principles of the theory," (1982, p4).

4.3 The interacting field vacuum

Note that the Fock space vacuum is the vacuum state of a free field, and, with respect to the Fock space number operator, a state of zero particles. The free field vacuum of Fock space is a distinct concept from what is often called the 'dressed vacuum' of interacting fields.

The dressed vacuum purportedly contains an infinite number of 'virtual' particles; particles which, if they existed, would violate the relativistic relationship $\langle p, p\rangle = -m^2$ between mass m and energy-momentum p.

The notion of virtual particles is often invoked to explain and justify otherwise ad-hoc renormalization procedures, which are used to obtain finite results from the perturbation series in the second-quantized scattering theory. However, the free-field vacuum in Fock space is the only vacuum which is theoretically well-defined in quantum field theory. As Prugovecki states, "the actual computations performed in perturbation theory actually begin with expressions for asymptotic states,... formulated in Fock space, and then progress through a

chain of computations dictated by Feynman rules, which have no direct bearing to a mathematically rigorous realization of a non-Fock representation of the canonical commutation relations... Hence, in conventional QFT [quantum field theory] the existence of such a representation, and of a corresponding unique and global 'dressed vacuum', is merely a conjecture rather than a mathematical fact," (1995, p198-199).

Rugh and Zinkernagel concur, arguing that the popular picture of the production and annihilation of virtual particles in the 'interacting' vacuum, "is actually misleading as no production or annihilation takes place in the vacuum. The point is rather that, in the ground state of the full interacting field system, the number of quanta (particles) for any of the fields is not well-defined. For instance, the photon number operator does not commute with the Hamiltonian for the interacting field system, hence one cannot speak of a definite number (e.g. zero) of photons in the vacuum of the full interacting system" (2002, Note 27.)

Aitchison attempts to spell out this point-of-view in greater detail, (1985, p352-353). He assumes that there is a free Hamiltonian H_0 and a full Hamiltonian of perturbative form $H = H_0 + H_{int}$, which are both defined on the same state-space. He postulates that there is a complete set of eigenstates $\{|A\rangle\}$ of the free Hamiltonian:

$$H_0|A\rangle = E_A|A\rangle \ ,$$

which are *not* eigenstates of the full Hamiltonian.[10] Each eigenstate $|\overline{A}\rangle$ of the full Hamiltonian is such that $(H_0 + H_{int})|\overline{A}\rangle = E_{\overline{A}}|\overline{A}\rangle$, and can be expanded in the basis provided by the energy eigenstates of the free Hamiltonian:

$$|\overline{A}\rangle = \sum_A c_A|A\rangle \ .$$

The interaction perturbs the state from an eigenstate of the free Hamiltonian into an eigenstate of the full Hamiltonian, which is a superposition of all the free Hamiltonian eigenstates, hence the system can subsequently transition into any one of those eigenstates; in effect, the interaction permits the transition between eigenstates of the free Hamiltonian with different energies. (In a later section, we will how this works in the case of the spontaneous emission of light).

Aitchison argues that the interacting vacuum is simply a special case of this logic. i.e., the ground state of the full Hamiltonian is different from the ground state of the free Hamiltonian, and the ground state of the full Hamiltonian can be expressed as a superposition of all the energy eigenstates of the free Hamiltonian. In this sense, the interacting vacuum contains an indefinite number of free particles, and in particular, the interacting vacuum of QED contains an indefinite number of photons. The photons from each definite particle-number state in the superposition are the virtual photons commonly attributed to the interacting vacuum.

[10] If the interaction term were 'trivial', i.e., of the form $H_{int} = \lambda I$, then the eigenstates of the free Hamiltonian would also be eigenstates of the full Hamiltonian, with an extra λ added to the eigenvalues. Hence, Aitchison assumes the interaction is non-trivial.

Aitchison acknowledges the assumption that the full Hamiltonian can be decomposed as a perturbation $H = H_0 + H_{int}$ of the free Hamiltonian, and points out that this condition will not hold in general. However, Haag's theorem raises a question-mark over whether the free-field vacuum and the interacting vacuum can *ever* exist in the same state-space.

Haag's theorem demonstrates that a free-field Fock space cannot directly represent an interacting field system.[11] It proves that a Fock space cannot possess the vacuum vector of a free-field and the vacuum vector of an interacting field.

A vacuum vector is required to be invariant under space-time translations, and a Fock space possesses, up to phase, a unique translation-invariant vector. The vacuum state $|0\rangle$ of a free field[12] must be the ground state of the free-field Hamiltonian $\widehat{H}_0$, in the sense that $\widehat{H}_0|0\rangle = 0$, and no vector in the ray spanned by $|0\rangle$, $\{c|0\rangle : |c| = 1\}$, can also be the ground state of the full Hamiltonian $\widehat{H}$ for an interacting system. If $\widehat{H}_0|0\rangle = 0$, as required, then the requirement that $\widehat{H}|0\rangle = 0$ cannot also be satisfied.

For example, in the case of a self-interacting scalar field, with an interaction Hamiltonian density $H_I(x) = \mathscr{P}(\phi(x)) = \phi^4(x)$, when field operators are substituted into this expression to obtain the interaction Hamiltonian density operator, it contains, at the very least, one term with four creation operators, which is not cancelled out by any other term. As a consequence, $\widehat{H}_I|0\rangle \neq 0$, and, in fact, $|0\rangle$ is not even an eigenstate of $\widehat{H}_I$, hence one cannot render $|0\rangle$ as the interaction vacuum by adding to $\widehat{H}_I$ a term containing a finite constant, (Fraser 2008).

Haag's theorem entails that the free-field vacuum and the interacting vacuum can only exist in the same state-space if one relaxes some of the conditions defining a vacuum state. It is still quite possible for the ground state of the free-field Hamiltonian and the ground-state of the full interacting Hamiltonian to exist in the same space if one relaxes the requirement of invariance under space-time translations. If one considers quantum field theory defined on a discrete spatial lattice of finite volume, then there are only a finite number of degrees of freedom (Duncan 2012, p369), hence there is a unique state-space up to unitary equivalence, so the free-field ground state and interacting ground state *have* to exist in the same state-space.

There are theorems in quantum field theory which can only be understood if such an approach is adopted. For example, the Gell-Mann-Low theorem assumes that there is a free-field vacuum $|0\rangle$ and an interacting vacuum $|\Omega\rangle$ in the same state-space, with a non-zero overlap $\langle\Omega, 0\rangle \neq 0$, and derives a relationship between the time-ordered products of interacting field operators

[11] See Earman and Fraser (2006) for an excellent discussion.

[12] Here we use $|0\rangle$ rather than $|\Omega\rangle$ to denote the *free-field* vacuum state, just in case there might also be the vacuum vector of an interacting field in the same Hilbert space.

and time-ordered products of free-field operators (Folland, p181-184):

$$\langle \Omega, T[\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)]\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0, T[\hat{\phi}_{01}(x_1)\cdots\hat{\phi}_{0n}(x_n)\exp(-i\int_{-T}^{T}\hat{H}_I(\tau)d\tau)]0\rangle}{\langle 0, T[\exp(-i\int_{-T}^{T}\hat{H}_I(\tau)d\tau)]0\rangle} ,$$

shown here for the case of a self-interacting scalar field.

In fact, there are circumstances where the use of the Gell-Mann-Low theorem can be justified without requiring a finite number of degrees of freedom. If the quantum field is confined to a cubical box of finite spatial volume $V = L^3$, then the spatial momenta $\mathbf{k}$ are restricted to a discrete lattice:

$$\mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^3 .$$

Duncan (2012, Section 10.5) expounds the case of a self-interacting scalar field of two different masses. The additional mass of the second field functions as a type of self-interaction potential term in the Hamiltonian. The vacuum of one $|0\rangle_1$ provides the free field vacuum, and the vacuum of the other $|0\rangle_2$ is effectively an interacting field vacuum. The restriction to discrete momenta permits one to define the overlap ${}_2\langle 0,0\rangle_1$ between the vacua of the two distinct fields. The Gell-Man-Low formula can then be used to define the propagator $i\Delta_F^{(2)}(x-y)$ of the 'interacting' field in terms of the free-field propagator $i\Delta_F^{(1)}(x-y)$. The infinite volume limit can then be restored, yielding a Poincare-invariant propagator $i\Delta_F^{(2)}(x-y)$ for the interacting field.

This is a particularly interesting type of case because a finite volume 'cut-off' corresponds to an infinite lattice of spatial momenta. There is no restriction on the magnitude of the spatial momenta, so no limit to the energy scale, and no lower-limit to the length-scale which is being 'probed'.

4.4 Do longitudinal photons exist?

In the quantization of the free electromagnetic field, the scalar photon states and longitudinal photon states are considered to be fictitious by-products of gauge-freedom. Once the set of gauge equivalence classes is taken, the only types of physical photons are identified to be transverse photons.

However, in the treatment of scattering processes in quantum electrodynamics, as outlined in the previous section, all four components $\widehat{A}_\mu$ of the electromagnetic potential appeared. In fact, longitudinal and scalar photons often re-surface in expositions of scattering between charged particles. They can purportedly be found in the photon propagator $D_F^{\mu\nu}(k)$.

Recall that, in Feynman diagram terms, the photon propagator

$$\langle \Omega | T[\widehat{A}_\mu(x)\widehat{A}_\nu(y)]\Omega\rangle = D_F^{\mu\nu}(x-y) ,$$

corresponds to the amplitude for the creation of a virtual photon at $\mathbf{x}$ and its annihilation at $\mathbf{y}$ if $y_0 > x_0$, and the amplitude for the creation of a virtual photon at $\mathbf{y}$ and its annihilation at $\mathbf{x}$ if $x_0 > y_0$, (Folland 2008, p147). The

expression for the photon propagator contains a gauge-fixing parameter ξ. When the so-called Feynman gauge $\xi = 1$ is chosen, the covariant momentum-space photon propagator becomes:

$$D_F^{\mu\nu}(k) = \frac{-\eta_{\mu\nu}}{k^2 + i\epsilon} \ ,$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor, and ϵ is a so-called 'positive infinitesimal'.

The momentum space photon propagator can be decomposed into a transverse component, a Coulomb component, and a physically obscure residual term, (Greiner and Reinhardt, p186-187):

$$D_F^{\mu\nu}(k) = D_{F(trans)}^{\mu\nu}(k) + D_{F(Coul)}^{\mu\nu}(k) + D_{F(resid)}^{\mu\nu}(k) \ .$$

The decomposition is both gauge-dependent, and Lorentz-frame dependent. Switching back to the configuration representation, and choosing a Lorentz frame defined by a timelike vector field $n = \partial_t$, the Coulomb part can be expressed as:

$$D_{F(Coul)}^{\mu\nu}(x - y) = \delta_{\mu 0}\delta_{\nu 0}\frac{\delta(x_0 - y_0)}{4\pi|\mathbf{x} - \mathbf{y}|} \ .$$

Hence, in the case where x and y are two points with the same timelike coordinates in the Lorentz frame of choice, there is an amplitude for the simultaneous creation of a virtual photon at $\mathbf{x}$ and its annihilation at $\mathbf{y}$, (or vice versa); this amplitude contributes to the instantaneous Coulomb interaction between charged particles in that Lorentz frame.

Greiner and Reinhardt conclude from this that "the Coulomb interaction arises from the combined exchange of longitudinal and scalar photons," (ibid. p187), and similar commitments to the existence of longitudinal and scalar photons can be easily found in the quantum field theory literature:

> "Longitudinal and scalar photons are not observed as free particles ... For an electromagnetic field in the presence of charges ... longitudinal and scalar photons play an important role as virtual particles in intermediate states and provide a covariant description of the instantaneous Coulomb interaction," (Mandl and Shaw, Section 5.2).

> "We know that the longitudinal part of the electric field results from a certain combination of first-order derivatives of the longitudinal and scalar potentials. Furthermore, the total longitudinal field energy is just the electrostatic Coulomb energy of the relevant system of point charges. Thus, the exchange of longitudinal and scalar photons between charges must correspond to the instantaneous Coulomb interaction between these," (Keller, p319).

However, a commitment to the existence of longitudinal and scalar photons, and their role in electrostatic forces, entails a commitment to the existence of

virtual photons. The virtual photons which propagate between pairs of vertices in Feynman diagrams are not constrained to follow null curves; in the process, they violate the special relativistic energy-momentum relation.

There are two possible interpretations here: either a static electric field is a state of the electromagnetic field in which there are no photons at all, or it is a state in which there are simply no *transverse* photons. If one accepts that longitudinal photons and scalar photons are simply a fictional by-product of gauge freedom prior to the imposition of constraints, then the only types of physical photons are transverse photons, and these only exist for free electromagnetic waves.

But if a static electric field contains no photons at all, then by what means are Coulomb forces exerted? In the absence of counteracting forces, an electrostatic field will impart momentum to a passing charged particle. How does this generation of momentum occur if there are no photons exchanged between the charged particles, and if action-at-a-distance is prohibited?

The problem can be cast in even starker terms if we consider an electrically charged black hole. Such an object is created by electrically charged material collapsing inside its Schwarzschild radius, or simply by electrically charged matter falling inside the event horizon of a pre-existing black hole. An electrostatic field is created in the region outside the event horizon of the black hole. The charged material inside the black hole is capable of exerting forces on charged particles outside the black hole by virtue of its electrostatic field. Yet the momentum cannot be generated by the exchange of transverse photons, because these cannot escape from inside the black hole, where the source of the field has fallen.

Another conceptual problem is that static electric fields often provide *classical* states of the electromagnetic field, in which the expectation value of the electric field is non-zero throughout a macroscopic volume of space, (for example, between the plates of a charged capacitor). If states such as these are to be represented on a photonic Fock space, it couldn't be done with a photon-number eigenstate $|n\rangle$, because all such states have $\langle n, \widehat{\mathbf{E}}(x)\, n\rangle = 0$. But it also couldn't be done with a coherent state, because such a state represents an oscillating system rather than a static system. And it couldn't be done with a superposition of coherent states because "a quantum superposition of distinct classical field configurations is outside the classical domain," (Field and Hughston, 1999).

This is a subtle issue, which is resolved only by re-assigning the longitudinal component of the quantized electric field to the state-space of the matter field. Consider the operator form of the expression which relates the electric field to the 4-vector potential:

$$\widehat{\mathbf{E}}(\mathbf{x},t) = -\nabla \widehat{A}_0(\mathbf{x},t) - \frac{\partial \widehat{\mathbf{A}}(\mathbf{x},t)}{\partial t}\ .$$

Let's follow the approach of Mari *et al* (2016), and Fourier transform the operators:

$$\widehat{\mathbf{E}}(\mathbf{k},t) = -i\mathbf{k} \widehat{A}_0(\mathbf{k},t) - \frac{\partial \widehat{\mathbf{A}}(\mathbf{k},t)}{\partial t}\ .$$

Assuming the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$, the operator form of the Gauss law is

$$\nabla \cdot \widehat{\mathbf{E}}^{\parallel}(\mathbf{x}, t) = \nabla^2 \widehat{A}_0(\mathbf{x}, t) = \widehat{\rho}(\mathbf{x}, t) \ ,$$

which Fourier transforms into:

$$k^2 \widehat{A}_0(\mathbf{k}, t) = \widehat{\rho}(\mathbf{k}, t) \ .$$

This entails that:

$$\widehat{A}_0(\mathbf{k}, t) = \frac{\widehat{\rho}(\mathbf{k}, t)}{|\mathbf{k}|^2} \ .$$

If we substitute this into the expression for the electric field operator we obtain:

$$\widehat{\mathbf{E}}(\mathbf{k}, t) = -\frac{i\mathbf{k}}{|\mathbf{k}|^2}\widehat{\rho}(\mathbf{k}, t) - \frac{\partial \widehat{\mathbf{A}}(\mathbf{k}, t)}{\partial t} \ .$$

For a time-independent electric field (i.e., an electrostatic field), the second term vanishes, and one obtains:

$$\widehat{\mathbf{E}}^{\parallel}(\mathbf{k}) = -\frac{i\mathbf{k}}{|\mathbf{k}|^2}\widehat{\rho}(\mathbf{k}) \ .$$

The operator $\widehat{\rho}(\mathbf{k})$ acts on the quantum state space of the matter field, hence the same is true of $\widehat{\mathbf{E}}^{\parallel}(\mathbf{k})$. Because the Gauss law is a condition on the longitudinal component of the electric field, it entails that the operator for the longitudinal component of the electric field is a function of $\widehat{\rho}$, hence it also is an operator on the state-space *of the matter field*.

In terms of the Fock space of quantum electrodynamics, $\mathscr{F}_\gamma \otimes \mathscr{F}_{e,e^+}$, the matter density operator has the form $I_\gamma \otimes \widehat{\rho}$, and the longitudinal component of the electric field is likewise of the form $I_\gamma \otimes \widehat{\mathbf{E}}^{\parallel}$, where I_γ is the identity operator on the photonic Fock space.

Thus, we have a component of the electromagnetic field which is represented by an operator on the state space of the fermionic matter system, rather than the bosonic state space of the free electromagnetic field.[13]

> "The longitudinal (i.e. proportional to $\mathbf{k}$) component of the electric field operator is determined by the charge-density operator, and acts on the Hilbert space of the particle alone. Therefore, even if the field is in its vacuum state, the expectation value of the electric field is the static Coulomb electric field generated by the expectation value of the charge density, and hence depends on the particle wave-function. This means that the state of the field alone does not contain all the information on the electric field, since its longitudinal component is encoded into the state of the particle.

[13] We will consider in a later section how coherent states of fermions can be treated as classical states. Hence, if the static electric field is determined by the degrees of freedom of charged fermions, then the static electric field associated with a charged fermion will be in a classical state if the fermion is.

> "Seen from a different perspective, the longitudinal component of the electric field is not a dynamical propagating degree of freedom, since it vanishes in absence of external charges and is completely determined by them, so there is no Hilbert space associated to it. The Hilbert space of the field contains only the degrees of freedom associated to the electromagnetic radiation, i.e. the magnetic field and the transverse (orthogonal to $\mathbf{k}$) component of the electric field. Then in a product state with the field part in the vacuum, only these components are in the vacuum mode, while there can be a static electric field depending on the state of the particle," (Mari *et al* 2016, p6-7).

A static electric field is associated with the vacuum state of the photonic Fock space, the zero-photon eigenstate. Whilst the photon is frequently proclaimed to be the gauge boson which mediates the electromagnetic force between charged particles, the Coulomb electrostatic interaction, upon which a large proportion of chemistry depends, is not explained by freely propagating photons.

Let's look at the role of the Gauss law constraint in this. When electromagnetism is formulated as a canonical theory, each point of the classical phase-space is pair of electric and magnetic fields, defined over 3-dimensional space $(\mathbf{E}(\mathbf{x}), \mathbf{B}(\mathbf{x}))$. There is a Cauchy problem in which the fields evolve in time $(\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t))$ given the specification of initial data $(\mathbf{E}(\mathbf{x}, 0), \mathbf{B}(\mathbf{x}, 0))$. In this canonical approach, the Gauss law $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}^{\parallel} = \rho$ is a constraint which applies at each moment of time. If the initial data $(\mathbf{E}(\mathbf{x}, 0), \mathbf{B}(\mathbf{x}, 0))$ satisfy the constraint, then it will be satisfied at every other moment of time.

The Gauss law, then, is a purely spacelike constraint on the longitudinal component of the electric field. It doesn't tell us how the longitudinal electric field evolves, it tells us what it is. As Earman puts it, "relatively spacelike electromagnetic events are mutually constrained by the laws of electromagnetism," (1995, p126).

In the context of a discussion of particle horizons in cosmology, Earman argues that an electrically charged particle, represented by a worldline γ passing through a space-time point p, can *feel* the tug of another charged particle, represented by a worldline δ, due to the Coulomb field, even if the causal past of p doesn't contain δ, $\mathscr{J}^{-}(p) \cap \delta = \emptyset$, and no signal has had time to propagate between the particles since the beginning of the universe, (ibid, p126-127).

In a canonical approach to the quantization of the electromagnetic field, one might impose the Gauss law constraint prior to quantization by restricting the phase-space to pairs which do satisfy the constraint. Alternatively, one can impose the constraint after quantization. This is also the natural approach when a covariant quantization scheme is used. One has a Gauss law operator

$$\widehat{G} = \nabla \cdot \widehat{\mathbf{E}} - \widehat{\rho}\,.$$

This is an operator on the state-space of the matter field. The physical states of the matter field ψ are required to satisfy the condition:

$$\widehat{G}\psi = 0\,.$$

This notion of a spacelike constraint supports the following interpretation: The overall concept of the quantum field theory programme is to reduce classical force fields to the exchange of gauge bosons. Disturbances in the electromagnetic field can be represented by the propagation of transverse photons, but the forces exerted by a static electric field cannot. The quantized electrostatic field extends through a region of space, and exerts forces locally on other charged particles. The longitudinal component of the electric field transported by a charged fermion, may not be an independent degree of freedom from the charged particle, but it cannot be equated with the charged particle either. The quantized static electric field is the intermediary which exerts forces on other charged particles.

If the quantum field theory programme to reduce classical force fields to the exchange of gauge bosons is to be complete, then static electric fields have to be reduced to the exchange of photons, the interaction carriers of the electromagnetic force. The notion that there are 'virtual' longitudinal and scalar photons is an unavoidable consequence of this. If the electrostatic force exerted at a point in space is to be exerted locally, and if the programme of reducing the electromagnetic field to the exchange of photons is to be implemented, then the local exertion of the force has to be attributed to the emission and/or absorption of photons at that point in space (or in a small bounded neighbourhood of it). For the field to be static, transverse photons are unacceptable, hence the only option is a structure which includes longitudinal and scalar photons.

Under this interpretation, one loses the capability to treat the exertion of electrostatic forces as a process occurring in time. Instead, the electrostatic field around a charged particle is reduced to a spacelike structure which includes longitudinal and scalar photons, bound to the electric charge.

In general, a quantized electromagnetic field $\widehat{\mathbf{E}}(\mathbf{x},t), \widehat{\mathbf{B}}(\mathbf{x},t)$ exerts a force on test particles in the neighbourhood of a point $\mathbf{x}$ in space, remote from the charged sources of the field, because the field can be decomposed into *spacetime* networks of creation, propagation, and annihilation processes involving photons (as well as the virtual quanta of the matter fields). The net effect of these processes is to impart momentum to a test particle, (in the absence of counteracting forces).

In particular, the electric field $\widehat{\mathbf{E}}^{\|}(\mathbf{x},t)$ associated with the Coulomb electrostatic force, exerts a force on test particles in the neighbourhood of a point $\mathbf{x}$ in space remote from the charged source of the field, because the field can be decomposed into *spacelike* networks involving longitudinal and scalar photons (as well as the virtual quanta of the matter fields). The net effect of these networks is to impart momentum to a test particle, (in the absence of counteracting forces).

Hence, while longitudinal and scalar photons cannot exist as free particles, they *do* exist as links in the spacelike networks into which the electrostatic interaction can be decomposed.

4.5 Spontaneous and stimulated emission of light

Let's consider one particular case of the interaction between radiation and matter, the emission of light. Photons can be emitted from excited atoms either by spontaneous emission or stimulated emission. Let's start with the case of spontaneous emission.

If an atom is treated in isolation from the electromagnetic field, then its energy eigenstates $\{\psi_{E_i} : i = 0, 1, 2 \ldots\}$, (i.e., the energy levels of its electrons, with being the ground state ψ_{E_0}) are all stationary states. Given the Hamiltonian for the atom H_{atom}, the energy eigenstates are all mutually orthogonal, and there is zero probability of a transition from one state to another, $\langle \psi_{E_i}, H_{atom} \psi_{E_j} \rangle = 0$.

However, if the full Hamiltonian of the interacting system is used, $H = H_{atom} + H_{rad} + H_{int}$, with

$$H_{int} = - \int \widehat{\mathbf{j}} \cdot \widehat{\mathbf{A}} \, d^3x \ ,$$

where $\widehat{\mathbf{j}}$ is the quantized charge-current density, and $\widehat{\mathbf{A}}$ is the quantized electromagnetic vector potential, then

$$\langle \psi_{E_i}, H_{int} \psi_{E_j} \rangle \neq 0 \ .$$

Hence, transitions between energy states of the atom are possible.

Let's represent this explicitly. Because the interacting system of atom and electromagnetic field has been introduced, the state space becomes a tensor product:

$$\mathscr{H}_{atom} \otimes \mathscr{H}_{rad} \ .$$

The states of the joint system in which a definite number of particles are present can be denoted as

$$\psi_{E_i} \otimes (n_{\mathbf{k}_1, \epsilon_1}, n_{\mathbf{k}_2, \epsilon_2}, \ldots) \ ,$$

in which ψ_{E_i} is the i-th energy level of the atom, and $(n_{\mathbf{k}_1, \epsilon_1}, n_{\mathbf{k}_2, \epsilon_2}, \ldots)$ denotes the number of photons of each mode and polarization.

The presence of the interaction term in the Hamiltonian entails that there are non-zero matrix elements of the form (Aitchison 1985, p345):

$$\langle \psi_{E_0} \otimes (1_{\mathbf{k}, \epsilon}, 0, \ldots), \widehat{\mathbf{A}}_{\mathbf{k}, \epsilon} \, \psi_{E_i} \otimes (0, 0, \ldots) \rangle \ ,$$

where $(0, 0, \ldots) = \Omega$ is the vacuum state of the free electromagnetic field, and ψ_{E_0} is the ground state of the atom. So not only is there a non-zero probability of a transition from an excited atomic state to the ground state, but in concert with this there is non-zero probability that the initial state of the electromagnetic field will transition from the vacuum state to a state with a photon present.

The spontaneous emission of photons from the excited atoms in a gas or solid is isotropic. This corresponds to an isotropic directional distribution of wave vectors $\mathbf{k}$. Moreover, spontaneously emitted light will also be distributed over a range of energies $E = \hbar c |\mathbf{k}|$:

"Normal atomic spontaneous emission takes place in a time $\Delta t \sim 10^{-9}s$, and the spontaneously radiated electromagnetic field has a frequency spectrum centred on the atomic transition frequency, with a width $\sim 1/\Delta t$ of about 100MHz. There are additional sources of frequency broadening of the emitted light spectrum. In a solid, there are crystal strains and inhomogeneities which distort the energy levels of the emitting ions. Ions in different parts of the crystal differ in their transition frequencies, so that light is emitted in a band of frequencies... In a gas, atoms move with a distribution of velocities ν, and radiate a field which is shifted in frequency by the Doppler effect by a fractional shift $(\Delta\omega/\omega) = \nu/c$. Atomic velocities are approximately 10^3 m s^{-1} so $\Delta\omega/\omega$ is about 10^{-5} and [for optical wavelength radiation of $\omega \sim 10^{15}$Hz] the Doppler shift $\Delta\omega \sim 10^{10}$Hz. Of course, there is a distribution of velocities and a range of emitted frequencies with a width approximately equal to $\Delta\omega$. The light from such atoms therefore has an extremely broad bandwidth," (Knight 1989, p291).

However, an excited atom can also be stimulated to emit photons by an incident flux of photons at the same frequency as the atomic transition. In the case of such 'stimulated' emission, the emitted radiation tends to be emitted in the same direction as the incident radiation, with the same frequency and polarization as the incident photons.

Laser light is produced by stimulated emission from a crystalline solid shaped in a rod-like geometry. A population of excited atoms is created, which initially emit photons spontaneously in all directions. Those which propagate along the axis of the rod will tend to stimulate the emission of more photons in the same direction, while those emitted in other directions will simply be lost. Hence, as the process continues, a population of photons with (almost) the same wave-vector $\mathbf{k}$ and polarization ϵ is created. Mirrors are typically placed at either end of the cavity in which the crystalline rod is suspended. This produces a beam of amplified light of a particular mode. Not only do the photons lie within a very narrow range of frequencies, but because they are concentrated in a particular direction, the energy density of the beam is much greater than that of isotropic radiation.

Laser light provides an approximate physical realisation of a 'coherent' quantum state of light, a concept to which we turn next.

5 Coherent states

It is sometimes claimed that laser light consists of a set of photons in the same state. In fact, if a laser is treated as a coherent state of the quantized radiation field, then this statement is, strictly speaking, false.

To understand why, first recall that according to quantum field theory, photons are merely the excitation quanta of the quantized free electromagnetic field. There are states of the quantized free electromagnetic field in which there are definite numbers of photons present, but there are also states in which there is an indefinite number of photons. Such states can be described as non-particle states of the quantized free electromagnetic field.

Let us begin by defining a 'coherent' state $|\alpha\rangle$ of light in terms of a system with a single mode. In this context a mode is defined by a particular combination $(\mathbf{k}, \epsilon)$ of wave-vector $\mathbf{k}$ and polarization ϵ. Expressed in the 'bra-ket' notation, a coherent state is defined in terms of a basis of photon number eigenstates as follows:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ ,$$

where $|n\rangle$ represents a state with n photons of wave-vector $\mathbf{k}$ and polarization ϵ. If $f_{\mathbf{k},\epsilon}$ represents the 1-particle state corresponding to the mode $(\mathbf{k}, \epsilon)$, then

$$|n\rangle = f_{\mathbf{k},\epsilon} \odot \cdots \odot f_{\mathbf{k},\epsilon} \quad \text{(n times)} \ .$$

The photon number eigenstates are also energy eigenstates, given that they represent states with n photons of energy $E = h\nu = \hbar c|\mathbf{k}|$. These photon number eigenstates are often dubbed 'Fock states', but given the potential for confusion with a general state in Fock space, we will avoid this usage.

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator $\widehat{a}$:

$$\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle \ .$$

In the case of a system with a single mode, $\widehat{a} = \widehat{a}(f_{\mathbf{k},\epsilon})$.

However, the annihilation operator is not self-adjoint, and the eigenvalue is a complex number $\alpha = |\alpha|e^{i\theta}$. This complex number is considered to specify the phase θ and amplitude $A = |\alpha|$ of the mode-$(\mathbf{k}, \epsilon)$ laser light, in the sense that the spatial distribution of expectation values for the electric field operator, resembles the form of a classical monochromatic wave of phase θ and amplitude A, (Duncan 2012, p229):

$$\langle \alpha, \widehat{\mathbf{E}}(\mathbf{x}, t)\alpha\rangle = -2CA \sin(\mathbf{k} \cdot \mathbf{x} - \omega t + \theta)\varepsilon \ ,$$

where $C = \sqrt{\hbar\omega_{\mathbf{k}}/2V}$ for a field confined to a cubical box of volume V, and ε is the real polarization vector associated with the mode $(\mathbf{k}, \epsilon)$. Hence, these

coherent states appear to be (semi-)classical states of the free electromagnetic field.[14]

To be an eigenstate of an annihilation operator is, in fact, only one of the criteria used to define coherent states. Along with Annihilation Operator Coherent States (AOCS), there are Displacement Operator Coherent States (DOCS), and Minimum Uncertainty Coherent States (MUCS). The coherent states of light satisfy all three criteria. In a later section we will encounter coherent states of fermions which are MUCS, but not AOCS.

Whilst the basis states $|n\rangle$ are eigenstates of the number operator, (i.e., interpretable as states in which a definite number of photons exist), a coherent state is clearly a superposition of these states, and is not itself an eigenstate of the number operator. Hence, there are an indefinite number of photons in a coherent state of light.

One way of putting it is that "a Fock state [i.e., a photon-number eigenstate] implies fluctuations in the field intensity, and a... coherent state implies fluctuations in the photon number," (Troup and Perlman, 1981), although it should be noted that even in a coherent state the field operators have a non-zero variance.

If one could measure the number of photons in a pulse of laser light, one would always measure a definite number, but one would measure different numbers on different occasions for an identical coherent state. Each particular coherent state has an expectation value for the number operator, which corresponds to the mean number of particles detected over repeated measurements. But a coherent state has non-zero dispersion for the number operator. Specifically, the probability of counting n photons of wave-vector $\mathbf{k}$ and polarization ϵ, is the probability $|\langle n, \alpha\rangle|^2$, which is given by a Poisson distribution:

$$|\langle n, \alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} = \frac{\mu^n}{n!} e^{-\mu} \, .$$

If one subscribes to an interpretation of quantum theory in which measurements trigger a non-unitary and non-deterministic collapse of the state, then $|\langle n, \alpha\rangle|^2$ is the probability that the coherent state will collapse into an n-photon eigenstate.

Textbook accounts of how optical detectors measure the amplitude of radiation all tend to *begin* with the absorption of a photon, the actual 'measurement-like interaction'. What follows is then an account of how that event is amplified up to the macroscopic level. For example, in CCDs, photomultipliers, and even in photographic emulsions, the chain of events begins with the photoelectric effect liberating an electron. (In the human retina, the absorption of a photon by a rod cell causes a subset of a pigment molecule to change from a *cis* to a *trans* configuration). Measurement of a pulse of coherent light in a state $|\alpha\rangle$ will generate n such events with probability $|\langle n, \alpha\rangle|^2$.

This Poisson distribution has a mean value of $\mu = |\alpha|^2$, and being a Poisson distribution, the variance equals the mean, and the standard deviation $\sigma =$

[14] Some authors use the term 'quasi-classical', and Duncan (p228) refers to coherent states as 'ultraclassical'. Pure classical states determine a precise dispersion-free value for every physical quantity, whereas coherent states suffer non-zero dispersion, so perhaps a more appropriate term would be 'infraclassical'.

$\mu^{1/2} = |\alpha|$. Hence, the greater the amplitude $|\alpha|$ of the complex number defining the coherent state, the greater the expected number of photons, but the relative uncertainty in the number of photons, given by the coefficient of variation σ/μ, is inversely proportional to the amplitude $|\alpha|$:

$$\frac{\sigma}{\mu} = \frac{|\alpha|}{|\alpha|^2} = |\alpha|^{-1} \ .$$

Just as the particle number is indefinite in a coherent state, so is the energy. But just like the particle number, the expectation value of the energy is finite. If such states of indefinite particle number and energy are deemed to be 'non-particle' states, then laser light is a non-particle state of the quantized free electromagnetic field.

The set of photon number eigenstates provide a countable, orthogonal basis for Fock space. The collection of all coherent states $\{|\alpha\rangle : \alpha \in \mathbb{C}^1\}$ is a non-denumerable set, whose members are neither linearly independent nor orthogonal. Specifically,

$$\langle \alpha, \beta \rangle = \exp(\alpha^* \beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) \ .$$

However, the set of coherent states do still span the Fock space. (They are said to form an 'overcomplete' family). The identity operator on Fock space can be decomposed either as a sum of projection operators on the photon-number eigensubspaces,

$$I = \sum_n \langle n, \cdot \, \rangle |n\rangle \ ,$$

or as an integral of projection operators onto the subspaces generated by the coherent-states:

$$I = \frac{1}{\pi} \int \langle \alpha, \cdot \, \rangle |\alpha\rangle \, d\alpha \ .$$

Hence, an arbitrary element $|f\rangle$ of Fock space can be expressed as:

$$|f\rangle = \frac{1}{\pi} \int \langle \alpha, f \rangle |\alpha\rangle \, d\alpha \ .$$

In particular, a photon-number eigenstate $|n\rangle$ can be expressed as an integral over all the coherent states:

$$|n\rangle = \frac{1}{\pi} \int \langle \alpha, n \rangle |\alpha\rangle \, d\alpha \ .$$

Note that because the coherent states are not linearly independent, any coherent state can be expressed as an integral over all the others:

$$\begin{aligned} |\alpha\rangle &= \frac{1}{\pi} \int \langle \beta, \alpha \rangle |\beta\rangle \, d\beta \\ &= \frac{1}{\pi} \int \exp(\beta^* \alpha - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) |\beta\rangle \, d\beta \ . \end{aligned}$$

So much for single-mode coherent states. To introduce the more general concept, consider first the 'wavefunctional' approach expounded in the opening section, in which elements of Fock space correspond to polynomial functions on the 1-particle Hilbert space $\mathscr{H}_\gamma$. In this approach, a coherent state is represented as an exponential function:

$$\phi_f(\psi) = e^{\langle \psi, f \rangle} \ ,$$

for some $f \in \mathscr{H}_\gamma$. If we recall that an exponential function can be expanded as a power series, then we obtain:

$$e^{\langle \psi, f \rangle} = 1 + \langle \psi, f \rangle + \frac{\langle \psi, f \rangle^2}{2!} + \frac{\langle \psi, f \rangle^3}{3!} + \cdots \ .$$

This maps to the following element of Fock space:

$$1 \oplus f \oplus \frac{(f \odot f)}{2!} \oplus \frac{(f \odot f \odot f)}{3!} \oplus \cdots \ ,$$

where the symmetric tensor product $\odot$ has been chosen for the bosonic case appropriate to photons.

Hence, seen from this perspective, a coherent state is the result of exponentiating the action of the creation operator $a^*(f)$ on Fock space. A coherent state, interpreted literally, is the consequence of an unlimited creation process. But because the process yields a normalized superposition over all the possible different particle numbers, the expectation value of the energy is finite.

In this more general approach, the coherent states in photonic Fock space are not indexed by complex numbers $\alpha \in \mathbb{C}^1$, as they were in the case of a system with a single possible mode, but by the 1-particle states $f \in \mathscr{H}_\gamma$. Honegger and Rieckers (2015, p436) name these 'Glauber vectors':

$$\phi_f = \exp(-\tfrac{1}{2}||f||^2) \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \odot_n f \ ,$$

One can verify as follows that these states are normalized:

$$\begin{aligned}
\langle \phi_f, \phi_f \rangle &= \exp(-||f||^2) \left\langle \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \odot_n f, \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \odot_n f \right\rangle \\
&= \exp(-||f||^2) \sum_{n=0}^{\infty} \frac{\langle f, f \rangle^n}{n!} \\
&= \exp(-||f||^2) \sum_{n=0}^{\infty} \frac{||f||^{2n}}{n!} \\
&= \exp(-||f||^2) \cdot \exp(||f||^2) \\
&= 1 \ .
\end{aligned}$$

The general inner product between two coherent states becomes:

$$\langle \phi_f, \phi_g \rangle = \exp(\langle f, g \rangle - \tfrac{1}{2}||f||^2 - \tfrac{1}{2}||g||^2) \ .$$

The vacuum vector Ω is a special coherent state, which corresponds to the zero vector $f = \mathbf{0} \in \mathscr{H}_\gamma$,

$$\phi_0 = \Omega \ ,$$

The Glauber vectors are duly eigenvectors of the Fock space annihilation operators:

$$a(f)\phi_g = \langle f, g \rangle \phi_g \ .$$

Note that just as there was no restriction to complex numbers α of unit modulus in the case of a system with only one possible mode, there is no restriction to elements $f \in \mathscr{H}_\gamma$ of unit norm.

A fact of keen philosophical significance is that a coherent state is not orthogonal to the vacuum state Ω. Specifically,

$$\langle \phi_f, \Omega \rangle = \exp(-\tfrac{1}{2}||f||^2) \ .$$

Hence, if the photon particle number of a coherent state is measured, there is always a non-zero probability $|\langle \phi_f, \Omega \rangle|^2 = \exp(-||f||^2)$, of measuring zero particles. The greater the norm of the vector f which indexes the coherent state, the lower the probability.

The fact that $\langle \phi_f, \Omega \rangle \neq 0$ for all coherent states ϕ_f, entails that if the initial state is the vacuum state, then there is a non-zero probability of a transition to any coherent state. This distinguishes the coherent states from the photon number eigenstates, which are orthogonal to the vacuum state.

If we accept that coherent states are classical states, then it follows that there is a non-zero probability of a transition from a vacuum state to a classical state. Once the transition to a classical state has occurred, then measurement-like interactions are capable of inducing a transition from that state into a photon-number eigenstate, but a direct transition from quantum vacuum to photon-number eigenstate is not possible.

Another way of looking at this is that the vacuum state Ω can be decomposed in a basis consisting of coherent states. We've already seen that in the case of a system with a single mode, any coherent state can be expressed as an integral over all the other coherent states. This result can be extended to the general case, but there are some technical complications.[15]

As a first step to surmounting these difficulties, consider an arbitrary m-dimensional subspace $\mathcal{V} \subset \mathscr{H}_\gamma$ of the 1-particle Hilbert space. There is a projection operator $\mathbb{P}_\mathcal{V} : \mathscr{H}_\gamma \to \mathcal{V}$ onto each such finite-dimensional subspace. In addition to the Fock space $\mathscr{F}(\mathscr{H}_\gamma)$ generated by the entire infinite-dimensional 1-particle space $\mathscr{H}_\gamma$, there is a Fock subspace $\mathscr{F}(\mathcal{V}) = \oplus_{n=0}^{\infty}(\odot_n \mathcal{V})$ for each finite-dimensional $\mathcal{V}$.

[15] I am much indebted to Alfred Rieckers for spelling out the following mathematical material.

Now suppose that we select an arbitrary orthonormal basis $\{e_k : k = 1, \ldots, m\}$ of $\mathcal{V}$, so that $f = \sum_{k=1}^{m} \alpha_k e_k$ for any $f \in \mathcal{V}$. The identity operator on $\mathscr{F}(\mathcal{V})$ can be decomposed as an integral over a finite-dimensional domain:

$$I_{\mathscr{F}(\mathcal{V})} = \frac{1}{\pi^m} \int_{\mathbb{R}^{2m}} \langle \alpha, \cdot \, \rangle |\alpha\rangle \, d^{2m}\alpha \ ,$$

where each $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$ now denotes the Glauber vector ϕ_f corresponding to $f = \sum_{k=1}^{m} \alpha_k e_k$.

We can write this as:

$$I_{\mathscr{F}(\mathcal{V})} = \int_{\mathcal{V}} \langle \phi_f, \cdot \, \rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \ ,$$

where $\lambda_{\mathcal{V}} = d^{2m}\alpha/\pi^m$ is the scaled Lebesgue measure on the Borel sets of the finite-dimensional space $\mathcal{V}$. This measure $\lambda_{\mathcal{V}}$ is independent of the choice of orthonormal basis.

One can define a projection operator $\Gamma(\mathbb{P}_{\mathcal{V}}) : \mathscr{F}(\mathscr{H}_\gamma) \to \mathscr{F}(\mathcal{V})$, such that for any $\psi \in \mathscr{F}(\mathscr{H}_\gamma)$:

$$\Gamma(\mathbb{P}_{\mathcal{V}})\psi = \int_{\mathcal{V}} \langle \phi_f, \psi \rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \ .$$

In particular, the vacuum vector Ω from the infinite-dimensional Fock space has a projection onto each $\mathscr{F}(\mathcal{V})$:

$$\Omega_{\mathcal{V}} = \int_{\mathcal{V}} \langle \phi_f, \Omega \rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \ .$$

The finite-dimensional subspaces $\mathcal{V}$ possess the structure of a directed set $\mathcal{I}$, as do the Fock subspaces $\mathscr{F}(\mathcal{V})$. (The inclusion relationship between the subspaces provides the binary $\geq$ relationship of this directed set). One can express the identity operator on the entire Fock space as the net limit, in the strong-operator topology,[16] of all the projection operators $\Gamma(\mathbb{P}_{\mathcal{V}})$:

$$I_{\mathscr{F}(\mathscr{H}_\gamma)} = \operatorname*{s-lim}_{\mathcal{V} \in \mathcal{I}} \int_{\mathcal{V}} \langle \phi_f, \cdot \, \rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \ .$$

One can then decompose an arbitrary $\psi \in \mathscr{F}(\mathscr{H}_\gamma)$ as

$$\psi = \lim_{\mathcal{V} \in \mathcal{I}} \int_{\mathcal{V}} \langle \phi_f, \psi \rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \ ,$$

where the limit is in the norm topology. Thus, a coherent state ϕ_g can be expressed as the limit of a net of integrals over the other coherent states:

$$\begin{aligned} \phi_g &= \lim_{\mathcal{V} \in \mathcal{I}} \int_{\mathcal{V}} \langle \phi_f, \phi_g \rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \\ &= \lim_{\mathcal{V} \in \mathcal{I}} \int_{\mathcal{V}} \exp(\langle f, g \rangle - \tfrac{1}{2}||f||^2 - \tfrac{1}{2}||g||^2) \phi_f \, d\lambda_{\mathcal{V}}(f) \ . \end{aligned}$$

[16] A net $(A_i)_{i \in \mathcal{I}}$ of bounded operators on a Hilbert space $\mathscr{H}$ converges strongly to A if $\lim A_i x = Ax$ for all $x \in \mathscr{H}$. In this event, every neighbourhood U of A is such that there is some $i_0 \in \mathcal{I}$ such that $A_i \in U$ whenever $i \geq i_0$.

In particular, one can decompose the vacuum vector Ω as

$$\begin{aligned}\Omega &= \lim_{\mathcal{V}\in\mathcal{I}} \int_{\mathcal{V}} \langle \phi_f, \Omega\rangle \phi_f \, d\lambda_{\mathcal{V}}(f) \\ &= \lim_{\mathcal{V}\in\mathcal{I}} \int_{\mathcal{V}} \exp(\langle f, \mathbf{0}\rangle - \tfrac{1}{2}||f||^2 - \tfrac{1}{2}||\mathbf{0}||^2)\phi_f \, d\lambda_{\mathcal{V}}(f) \\ &= \lim_{\mathcal{V}\in\mathcal{I}} \int_{\mathcal{V}} \exp(-\tfrac{1}{2}||f||^2)\phi_f \, d\lambda_{\mathcal{V}}(f) \; .\end{aligned}$$

6 Why is there something classical rather than nothing classical?

6.1 Coherent states and the vacuum

If one were to invoke the many-worlds interpretation of quantum theory, then each term of $\Omega = \lim_{\mathcal{V}\in\mathcal{I}} \int_{\mathcal{V}} \exp(-\frac{1}{2}||f||^2)\phi_f \, d\lambda_{\mathcal{V}}(f)$ would correspond to a different 'branch' of the universe. Because each term corresponds to a different coherent state, each branch is classical, yet the overall state is the quantum vacuum.

If there is a quantum field theory for the universe as a whole, so that there is a well-defined vacuum state for the entire universe, and if there are coherent states for the entire universe which span the state-space, then this logic could be generalised from the quantized radiation field to the universe as a whole. If so, it would offer a joint explanation for why the world appears to be classical, and why there is something (classical) rather than nothing (classical).[17]

Indeed, there is at least one precedent for such a proposal in the philosophy of physics literature:

> "The relativistic vacua... are eigenstates only of *global* observables (observables like the total energy, the total charge, [the particle number] and so on) and not of any local ones.
>
> "States which entail, say, that there is a table across the room, and states wherein the world appears roughly as it appears to us (full of approximately localized objects, full of systems which are changing with time), are *not* necessarily orthogonal to [relativistic] vacua... observers such as ourselves cannot establish, by any practical means, that our experience is not merely a constituent, merely a *branch*, of that vacuum," (Albert 1988, p128-129).

Needless to say, the quantum vacuum cannot be equated with nothing; it is merely the quantum field state in which there are no particle-like excitations. Even if the idea worked, one would still be left without an explanation of why there is a quantum vacuum rather than nothing. Moreover, for the idea to actually work, there are at least two technical hurdles which need to be cleared:

1. Fermions cannot possess coherent states in the sense that bosons can.

2. There is no quantum theory of gravity.

We next consider these problems in turn.

[17] Generalising from the quantized radiation field, one might be tempted to declare 'Let there be light!'.

6.2 Fermions, Supersymmetry and Grand Unified Theories

The first problem is clearly defined by Weingard:

> "It is... true that Fermi fields can exhibit classical, macroscopic behaviour. The He^3 superfluid and superconductivity are two examples. But the basic particles of these phenomena are composite bosons. In elementary particle physics (and chemistry), however, what we see are not composite bosons, but fermions engaging in the basic interactions. These are the quanta of fermion fields which do not admit of 'macroscopic' coherent states," (1991, p215).

Now, as we will see in the final section, there *are* fermion states widely referred to as 'coherent'. These are Minimum Uncertainty Coherent States (MUCS) rather than Annihilation Operator Coherent States (AOCS). Specifically, in the non-relativistic quantum mechanics of a simple harmonic oscillator, there are fermion states which minimise the product of the position and momentum variance, and whose mean values follow the classical equations of motion. These states form an overcomplete family, spanning the state-space of the harmonic oscillator, (Gazeau 2009, p4).

However, in quantum field theory the problem is that the fermions in a composite system cannot exist in the same state, by virtue of the Pauli exclusion principle. This fact is encoded in the canonical anti-commutation relations for fermion creation operators:

$$b^*(f)b^*(g) + b^*(g)b^*(f) = 0 \, .$$

It follows that

$$b^*(f)b^*(f) + b^*(f)b^*(f) = 0 \, ,$$

which entails that $b^*(f)^2 = 0$. Hence, $b^*(f)|0\rangle = |1\rangle$, but $b^*(f)|1\rangle = b^*(f)^2|0\rangle = 0$. For any fermion mode, the occupation number is either zero or one. Hence, apart from $\alpha|0\rangle + \beta|1\rangle$, there cannot be coherent fermion states with an indefinite number of fermions of the same mode.[18]The consequence is that fermionic coherent states do not span the state-space, and in particular we cannot decompose the vacuum vector of the fermionic matter fields as a sum of coherent states.

One might wonder if it is possible to use Supersymmetry to mitigate this problem, given that the supersymmetry 'supercharge' operator Q transforms bosons into their fermionic partners and vice versa. Recall that Supersymmetry represents the state space of each system as a direct sum of a bosonic space and a fermionic space. The supercharge operator Q can be decomposed as $Q = q + q^*$,

[18]Note that 'Grassmann coherent states' have been defined for fermions. These are eigenstates of the fermion annihilation operators with Grassmann eigenvalues. The Grassmann eigenvalues, however, lack a physical interpretation.

where q annihilates a fermion and creates a boson, while q^* annihilates a boson and creates a fermion. Baker (2018) provides a simple example for a system in which there are no momenta, so that particles can only contribute their mass to the total energy: if (a, a^*) denote the boson annihilation and creation operators, and (b, b^*) denote the fermionic counterparts for particles of the same mass m, then:

$$q = \sqrt{(}m)a^*b \ ,$$

annihilates a fermion and creates a boson, while

$$q^* = \sqrt{(}m)b^*a \ ,$$

annihilates a boson and creates a fermion, without changing the total energy of the system.

Unfortunately, as Baker notes, only one boson of a particular mode can be swapped for a fermion. "$(q^*)^2 = 0$, which means that we cannot violate the exclusion principle by replacing multiple bosons with fermions." Conversely, the coherent boson states which span the bosonic state space are not the image of a fermionic state under the action of q. So even supersymmetry cannot eradicate the special status of bosonic coherent states.

Despite this shortfall, is it possible to devise a universe creation scenario using just the properties of the bosonic vacuum? There may be an opportunity to do just that, because according to certain Grand Unified Theories (GUTs), the massive fermions of the Standard Model are the decay products of massive gauge bosons. Hence, it could be postulated that the initial state of the universe was the vacuum state of a bosonic field.

A universe creation scenario might proceed by postulating the vacuum state of gravity Ω_{G}, tensored with the vacuum state of all the other matter fields and gauge force fields:

$$\Omega = \Omega_{\mathrm{G}} \otimes \Omega_{\mathrm{GUT}} \ .$$

We will consider what Ω_{G} might be in the next section. For the sake of argument, the vacuum state of the non-gravitational fields is chosen to be Ω_{GUT}, the vacuum state of a non-supersymmetric Grand Unified Theory. In schematic terms, a coherent state decomposition of Ω_{G} entails:

$$\begin{aligned}
\Omega &= \Omega_{\mathrm{G}} \otimes \Omega_{\mathrm{GUT}} \\
&= \lim_{\mathcal{V} \in \mathcal{I}} \int_{\mathcal{V} \subset \mathscr{H}_{\mathrm{G}}} c(g)\, \phi_g \, d\lambda(g) \otimes \Omega_{\mathrm{GUT}} \\
&= \lim_{\mathcal{V} \in \mathcal{I}} \int_{\mathcal{V} \subset \mathscr{H}_{\mathrm{G}}} c(g)\, \phi_g \otimes \Omega_{\mathrm{GUT}} \, d\lambda(g) \ ,
\end{aligned}$$

where $c(g)$ are the complex coefficients. The global vacuum state would decompose into a linear combination of every possible classical coherent state ϕ_g of 3-dimensional space, tensored with the vacuum state Ω_{GUT} of all the non-gravitational fields.

The latter could then be split into the non-gravitational bosonic and fermionic degrees of freedom:

$$\Omega_{\rm GUT} = \Omega_{\rm bos} \otimes \Omega_{\rm ferm} \ .$$

A coherent state decomposition of the bosonic vacuum would then entail:

$$\begin{aligned}\Omega_{\rm GUT} &= \Omega_{\rm bos} \otimes \Omega_{\rm ferm} \\ &= \lim_{\mathcal{V}\in\mathcal{I}} \int_{\mathcal{V}\subset\mathscr{H}_{\rm bos}} \exp(-\tfrac{1}{2}||f||^2)\, \phi_f \, d\lambda(f) \otimes \Omega_{\rm ferm} \\ &= \lim_{\mathcal{V}\in\mathcal{I}} \int_{\mathcal{V}\subset\mathscr{H}_{\rm bos}} \exp(-\tfrac{1}{2}||f||^2)\, \phi_f \otimes \Omega_{\rm ferm} \, d\lambda(f) \ .\end{aligned}$$

Now assume that we have a GUT which predicts the existence of X-bosons. We define an X-boson in generic terms as a GUT gauge boson which decays into quarks and leptons. In each classical coherent branch of this primordial vacuum state there would be a non-zero expectation value for the number of X-bosons per unit volume. Because these GUT bosons decay into quarks and leptons, all the particles of familiar big-bang cosmology would then be available.

Unfortunately, even this would not be sufficient to provide a starting point for big-bang cosmology. For that, one needs: (i) a plasma of quarks, leptons and gauge bosons to be in a high temperature thermal state; and (ii) expansion of the spatial geometry.

Without the expansion of space, the thermal state, as an equilibrium state, would be time-invariant. Only with the expansion of space can the material system be driven out thermal equilibrium, setting the scene for the subsequent formation of luminous stars and galaxies, and the far-from-equilibrium open systems they support.

6.3 Gravity and gravitons

One considerable hurdle to a well-defined universe creation scenario is the absence of a theory of quantum gravity. As Manin notes, "in any cosmogenic scenario a description of the origin of matter must be accompanied by a description of the origin of four-dimensional space-time," (1988, p6). For the scheme outlined in the preceding section to be well-defined, one would need to include gravity, as well as all the other forces and matter fields, in the definition of a global vacuum state. Nevertheless, despite the absence of a theory of quantum gravity, there *is* a well-defined Fock space for the graviton (Ashtekar and Geroch, 1974, p1250-1252), the hypothetical quantum mode of excitation of the *free* gravitational field, so let's see if anything can be gleaned from this.

The restriction to the free gravitational field is equivalent to a restriction to regions of space-time in which the Weyl tensor is non-zero, but the Ricci tensor vanishes. In particular, the gravitational Fock space is constructed from the space of linearized Weyl tensors $C_{\mu\nu\xi o}$ on flat Minkowski space-time which satisfy the Bianchi identities. Equivalently, one can work directly with the space

of gauge equivalence classes of metric tensors $\gamma_{\mu\nu}$ which yield such Weyl tensors. (In this context, the Weyl tensor is the analogue of the free electromagnetic field $F_{\mu\nu}$ from Maxwellian electromagnetism, and the metric tensor is analogous to the electromagnetic vector potential A_μ). This linear space of real tensor fields corresponds to a classical mass zero, spin-two field. It represents a space of perturbations to the flat metric on Minkowski space-time.

The one-particle space for the graviton is constructed by turning this real vector space into a complex Hilbert space $\mathscr{H}_{\rm G}$. The bosonic gravitational Fock space is then:

$$\mathscr{F}_{\rm G} = \bigoplus_{n=0}^{\infty} \mathscr{H}_{\rm G}^{\odot_n} \ .$$

This gravitonic Fock space is equipped with a well-defined, unique, Poincare-invariant vacuum vector $\Omega_{\rm G}$, along with gravitonic creation and annihilation operators indexed by momentum and helicity, and field operators constructed from them.

At first sight, scattering interactions between gravitons and the quanta of matter fields, such as the Dirac field for electrons and positrons, could be calculated by introducing a tensor product Fock space,

$$\mathscr{F}_G \otimes \mathscr{F}_{e,e^+} \ ,$$

and by the definition of an interaction Hamiltonian on this space, which couples the gravitational field to the matter field via the quantized stress-energy tensor. As is well-known, the normal calculational scams of quantum-field theory prove to be 'perturbatively non-renormalizable' in this case.

Nevertheless, in the absence of an adequate theory of quantum gravity, the gravitonic Fock space is important as a conceptual guide. As a bosonic system, there is an expectation that the quantized free gravitational field will possess coherent states, which correspond to classical states of the free gravitational field, i.e., gravitational waves. If one proceeded naively, one would write these states as:

$$\phi_e = \exp(-\tfrac{1}{2}||e||^2) \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \odot_n e \ ,$$

where $e \in \mathscr{H}_{\rm G}$ is any 1-graviton state.

If these coherent states are relevantly similar to those of the free electromagnetic field, they will span the state space, and provide a decomposition of the vacuum vector $\Omega_{\rm G}$. Hence, the state of zero gravitons could be decomposed into branches, each one of which corresponds to an excited classical state of the gravitational field.

However, this particular vacuum state of the gravitational field, $\Omega_{\rm G}$, simply corresponds to an absence of free field gravitational perturbations on a flat Minkowski background. That background is still there in the vacuum state, defining the geometry of space-time. So $\Omega_{\rm G}$ is *not* a state which defines an absence of space-time geometry.

Nevertheless, if a state in which there are no gravitational perturbations could be decomposed into classical branches, each of which is a well-defined coherent state of the free gravitational field, it would be highly suggestive, and one might be tempted by the following hypothesis:

> "The geometry of space-time is a classical, macroscopic structure. It can (presumably) be given a quantum mechanical account because Bose fields have coherent states, whose expectation values are the classical quantities," (Weingard 1991, p215).

It's important to remember, however, that the coherent states of the gravitational field and the electromagnetic field, defined in quantum field theoretic Fock space, are only classical states of the respective *free* fields. The classical geometry of space-time won't be captured by states of the free quantized gravitational field.

Recall that the longitudinal component of the electric field, determined by the charge density, has to be defined as an operator on the state-space of the matter field rather than the state space of the free electromagnetic field. The same logic applies to the longitudinal component of the gravitational field, determined by the mass density.

The 'weak' gravitational field of a planet or main-sequence star can be represented by a perturbation to the flat Minkowski space-time background. Just like a gravitational wave, this is represented by a perturbation to the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ .$$

However, this perturbation has a non-zero Ricci tensor. A general perturbation will satisfy the linearized Einstein equation (Le Tiec and Novak, 2016, p26):

$$\Box \overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ ,$$

where $\overline{h}_{\mu\nu}$ is a modified form of $h_{\mu\nu}$ with a trace of the opposite sign, and the Lorenz gauge $\partial^{\nu}\bar{h}_{\mu\nu} = 0$ has been used. This is clearly a wave-equation for the metric, in which the stress-energy-momentum $T_{\mu\nu}$ is the source. As noted above, freely propagating gravitational waves are obtained when the Ricci tensor vanishes, which is equivalent to the vanishing of the stress-energy-momentum tensor. Hence, freely propagating gravitational waves are solutions of:

$$\Box \overline{h}_{\mu\nu} = 0 \ .$$

In the case of a weak gravitational field generated by a massive body, $T_{\mu\nu} \neq 0$. The Newtonian gravitational potential Φ_G is incorporated into the g_{00} component of the metric by the definition, (Frankel, p28)

$$-\Phi_G = 1 - \sqrt{-g_{00}(r)} \ .$$

Assuming $|\Phi_G| \ll 1$, $g_{00} = -(1 + 2\Phi_G)$. This enables us to reproduce the Poisson equation of Newtonian gravitation:

$$\nabla^2 \Phi_G = \nabla^2 \sqrt{-g_{00}} = 4\pi G \rho_m \ ,$$

where ρ_m is the mass density. Now, $\nabla^2\sqrt{-g_{00}} = -R^0_0\sqrt{-g_{00}}$, where $R_{\mu\nu}$ is the Ricci tensor, hence we have an equation which links the Ricci tensor to the mass density, (ibid. p29):

$$R^0_0\sqrt{-g_{00}} = -4\pi G\rho_m \ .$$

This is the weak field gravitational equivalent of the Gauss law constraint. Once the matter degrees of freedom are quantized, a field operator $\widehat{\rho}_m$ has to be substituted for the mass density, and the quantized product of $R^0_0\sqrt{-g_{00}}$ is simply a function of this, an operator on the state space of the matter field.

One can refer to the product $R^0_0\sqrt{-g_{00}}$ as a longitudinal degree of freedom, in analogy with the longitudinal electric field. In the purely Newtonian theory, the Poisson equation for the gravitational potential $\nabla^2\Phi_G = 4\pi G\rho_m$ entails that the gravitational field of Newtonian theory, $\mathbf{g} = -\nabla\Phi_G$, satisfies $\nabla\cdot\mathbf{g} = -4\pi G\rho_m$. From the fact that $\mathbf{g} = -\nabla\Phi_G$, it follows that Newtonian gravity is a 'conservative force', so that $\nabla\times\mathbf{g} = 0$. In other words, the Newtonian gravitational force field is a longitudinal vector field.

Gravitational waves are transverse shear waves in the geometry of space. If they can be successfully quantized, they will be represented by coherent states of transverse gravitons. The static gravitational force fields around massive bodies, however, are analogous to the Coulomb fields of electromagnetism, and correspond to longitudinal degrees of freedom.

As with electromagnetism, there is then a judgement to be made over whether the longitudinal degrees of freedom should be relegated to functions of the matter field, or whether longitudinal gravitons actually exist. Here is one particularly clear expression of the latter point-of-view:

> "Electromagnetic radiation contains transverse (observable) photons which transport energy, the Coulomb field contains longitudinally polarized (virtual) photons which do not carry energy away and cannot be observed as free particles. Hence, there is a direct interaction between a transverse photon and a gravitational field of a black hole, but no gravitational interaction between a longitudinal photon and black hole. Hence, the Coulomb field (electrostatic interaction by 'lines of force' and Gauss' law) is able to cross the event horizon of a black hole. Within the linearized theory of gravity, a similar distinction between longitudinal and transverse gravitons can be made. Thus, a black hole can attract gravitationally matter and radiation outside its event horizon, because the Newtonian gravitational field is caused by longitudinal gravitons," (Argyris and Ciubotariu 1998, p149).

Certainly, a universe creation scenario of the type considered here will have to make the leap of assuming that the macroscopic geometry of 3-dimensional space corresponds to a coherent state in a yet-to-be discovered theory of quantum gravity. A vacuum state of the gravitational field in this generalised sense would be decomposable as a combination of all the possible coherent states, not merely those corresponding to gravitational waves.

6.4 Inflation

Before moving on, let's consider, for the sake of comparison, the type of vacuum decay envisaged by the inflationary cosmology scenario. This postulates the existence of self-interacting scalar field ϕ, minimally coupled to gravity, called the 'inflaton'. The scalar field corresponds to an effective perfect fluid in which the potential $V(\phi)$ determines the equation-of-state $p = w\rho$. The effective pressure of the scalar field is given by:

$$p = \dot{\phi}/2 - V(\phi) - (\nabla\phi)^2/6R^2 \ ,$$

where R is the scalar curvature. The energy density is:

$$\rho = \dot{\phi}/2 + V(\phi) + (\nabla\phi)^2/2R^2 \ .$$

If the spatial derivatives are dominant, then

$$w = \frac{p}{\rho} \approx \frac{-(\nabla\phi)^2/6R^2}{(\nabla\phi)^2/2R^2} = -\frac{1}{3} \ ,$$

which would drive an expanding universe, but one with $R(t) \propto t$, rather than the exponential expansion required for the inflationary scenario (Kolb and Turner, p277).

If instead the spatial derivatives are assumed be zero (or negligible), and the self-interaction is zero (or negligible) $V(\phi) = 0$, then

$$w = \frac{p}{\rho} = \frac{\dot{\phi}/2}{\dot{\phi}/2} = 1 \ ,$$

which would also fail to yield inflation.

It is only when the spatial derivatives are assumed to be zero, and the time-derivative is assumed to be negligible (as it must be for 'slow-roll' inflation), and the self-interaction is non-negligible, that we arrive at the equation-of-state for an effective Cosmological Constant:

$$w = \frac{p}{\rho} = \frac{-V(\phi)}{+V(\phi)} = -1 \ .$$

This will drive the exponential expansion required by the inflationary scenario.

To provide the initial conditions for big-bang cosmology, inflation postulates that this very specific self-interacting scalar field was subject to two types of decay:

1. A 'false vacuum' state decays to a 'true vacuum'. The true vacuum corresponds to the global minimum of the classical potential energy function. During the decay of the false vacuum, the universe undergoes exponential expansion at an approximately constant energy density.

2. At the end of inflation, the inflaton field oscillates about its global minimum. These oscillations correspond to the existence of scalar particles, the quanta of the inflaton field. These scalar particles decay into other, lighter particles, and the energy of the inflaton field oscillations 'reheats' the universe into a high temperature plasma appropriate as the starting point for big-bang cosmology.

There are two immediate problems with the concept of the true vacuum here. The true vacuum is the vacuum state of a self-interacting scalar field, hence it will be the vacuum state of the full interaction Hamiltonian in the quantum field theory. This is problematic, because the interaction Hamiltonian of a self-interacting scalar field doesn't commute with the particle number operator. Hence, the true vacuum state of the inflaton cannot be a state with a definite number of particles. The second problem is that the universe is in a state of expansion, and the existence of a unique vacuum state requires a stationary space-time. Hence, the ground state of the full interaction Hamiltonian will change from one moment of time to the next.

Over and above these problems with the concept of the true vacuum, we have to consider the postulated oscillations of the inflaton field. These oscillations are considered to be coherent states of a bosonic field: "When the scalar field ϕ reached the steep part of the potential, it fell quickly to the minimum and then oscillated about it. These oscillations were damped by terms in the equations of motion which arise due to the coupling of ϕ to the other fields in the theory... Note that the oscillations of ϕ describe a coherent state of zero momentum scalar particles - the momentum is zero because the scalar field is homogeneous. The damping of these oscillations may therefore be interpreted as the decay of the scalar particles into lighter species," (Blau and Guth 1987, p560).

As we have explored in detail above, a coherent state of a quantum field is not an eigenstate of the particle-number operator; it is a state with an indefinite number of particles. Hence, the decay of the coherent inflaton state into a post-inflationary plasma with a definite number of particles would requires some form of wave-function collapse or decoherence. In the case of decoherence, it is unclear what could provide the 'environment' or 'macroscopic degrees of freedom' which become entangled with the coherent degrees of freedom.

In the absence of an empirically adequate Grand Unified Theory which explains the apparent existence of Cold Dark Matter, as well as all the familiar particles and fields of the Standard Model, inflationary cosmology tends to be non-specific about the immediate decay products of the inflaton field oscillations. The particles observable in the universe today are considered to be merely the end of a decay chain which started with the decay of the inflaton quanta.

In the period of time between the end of inflation and the time of reheating, the mass-energy density of the universe was dominated by the coherent ϕ oscillations, (Kolb and Turner 1990, p279). At the end of inflation, the fields into which the inflaton particles decay can be thought of as being in their vacuum state, Ω_{decay}. The interaction of the inflaton field with these other fields causes

the decay of the coherent inflaton state, and the transition from Ω_{decay} to non-vacuum states which ultimately yield the quarks, leptons and gauge bosons of the Standard Model.

However, as alluded to above, the existence of a unique vacuum state requires a stationary space-time. Hence, even after the termination of the exponential expansion, the oscillations of the inflaton ϕ make it difficult to define a unique vacuum state for any field. In addition, there are competing requirements placed upon the strength of the coupling between the inflaton and the other fields: it has to be very small to facilitate an almost-flat potential energy function, appropriate for 'slow-roll' inflation; yet it has to be large enough to permit the reheating required for consistency with big-bang cosmology.[19]

In summary, inflationary cosmology is able to produce the starting conditions for big-bang cosmology because it combines the decay of a coherent bosonic state with the prior decay of a false vacuum state, to yield both a high temperature thermal state and an expanding geometry.[20]

To reiterate, a universe creation scenario which proposes a *decay* of the initial vacuum state, or the *decay* of a coherent state, is not the same thing as suggesting that the current state of the universe *is* the vacuum. Hence, the inclusion of fermionic degrees of freedom, and the need for an expanding geometry, seems to force a universe creation scenario away from the notion that the global state of the universe could be a vacuum state, and towards the more familiar idea from modern cosmology that some sort of vacuum decay transpired.

[19] Note that whilst most inflationary scenarios assume the lifetime of the coherent inflaton oscillations to be extremely small compared to the current age of the universe, there are variations on the theme which propose that the coherent ϕ-oscillations have only partially decayed, and the mass-energy density of the remainder is masquerading as Cold Dark Matter.

[20] There are, of course, other important putative successes of inflationary cosmology, such as providing the spectrum of density-perturbations which seeded galaxy-formation, and 'solving' the so-called horizon and flatness problems.

7 Coherent states and the classical world

7.1 Are coherent states classical?

Coherent states are often presented as the class of pure quantum states which most resemble the pure classical states: "A coherent state... is the closest quantum analog to a classical electromagnetic wave. While the field is not assigned a definite (vector) magnitude anywhere, the expected result of measuring it varies from point to point in the same way as the magnitude of a classical field, while individual measurement results are subject to random fluctuations that decrease in relative proportion to this expected magnitude. Measurements would also reveal a pretty well-defined phase at each point," (Healey, 2013).

However, the claim that the coherent states of the radiation field are classical states has several potential problems and questions to answer:

1. The classical states are not emergent from the quantum states.
2. Do the mixed classical states have unique decompositions?
3. The classical states are not obtained in the limit where there is a large number of photons.
4. The classical states are not reference-frame independent.

Let's elucidate each one of these in turn, starting with the implication that classical states are not emergent from quantum states.

When we look at Fock space, we have one basis which consists of classical states (i.e., coherent states), and another basis which consists of photon-number/energy-eigenstates. Any coherent state can be expressed as a linear combination of photon-number eigenstates, and any photon-number eigenstate can be expressed as a linear combination of the coherent states.

> "A general (pure) state of the quantized electromagnetic field... has a decomposition as a superposition of [photon-number] basis states, and so a typical state vector is associated with an indefinite number of photons... An arbitrary (pure) state also has a decomposition as a superposition of coherent states, each of which approximates a classical electromagnetic wave. It may therefore be thought of either as composed of a (typically indefinite) number of photons from various modes, or as partially constituted by each of a continuous range of distinct, approximately classical, electromagnetic waves. Since neither decomposition has priority, neither tells one what light represented by such a state is really composed of: a photon-number basis state may be decomposed into coherent states just as a coherent state can be decomposed into photon-number basis states," (Healey 2013).

If the radiation field is in a classical coherent state, then an appropriate interaction to measure the number of photons in that state will induce a transition into a quantum state with a definite number of photons. Conversely, if the

radiation field is in a quantum state with a definite number of particles, then an appropriate measurement-like interaction will induce a transition into a classical coherent state. Far from yielding or defining the classical world, measurement-like interactions can switch the state back and forth between the classical and the quantum.

Furthermore, because coherent states are not orthogonal, repeated measurements can induce transitions from one coherent state to another. If coherent states are classical states, they are classical states which are unstable under measurement-like interactions.

In defiance, then, of much of the literature exploring the relationship between the classical world and the quantum world, the classical is not emergent here. If we accept that photon-number eigenstates are quintessentially quantum, and coherent states are classical, then the quantum and the classical exist here at the same level.

In fact, one can flip the concept of classical emergence on its head, and portray the quantum world as emergent from the interference of classical worlds. This is perhaps clearest seen in the path-integral formulation of a quantum theory, where the propagator $K(q,t;q_0,t_0)$ which determines the evolution of the quantum state,

$$\Psi(q,t) = \int_{\mathscr{C}} K(q,t;q_0,t_0)\Psi(q_0,t_0)\,dq_0 \ ,$$

is decomposed as a sum (or integral) over kinematically-possible classical paths or histories:

$$K(q,t;q_0,t_0) = \int_{\mathscr{P}} e^{iA(\gamma)/\hbar} d\mu(\gamma) \ .$$

The action $A(\gamma)$ is a functional defined on the space of paths γ which interpolate between the configuration q_0 at time t_0, and the configuration q at time t.[21]

Orthogonal branches of the quantum state correspond to mutually disjoint bunches of interfering classical histories. If the expression for the classical action is altered, then the quantum state also changes, so the evolution of the quantum state supervenes on the kinematically-possible classical histories.

Amongst the space of kinematically possible histories which interpolate between the configuration q_0 at time t_0, and the configuration q at time t, will be one history which is a critical point of the action-functional. This is the single dynamically-possible classical history; i.e., it solves the classical dynamical equations. Quantum states whose mean values track this critical-point of the classical action can be said to be classical (or 'semi-classical', or 'quasi-classical') states.

Hence, in this sense, classical theory can provide both the substrate and the superstrate of quantum theory. Even the purported classical states, however, are the result of interfering bunches of classical histories. So, whilst there is certainly a requirement to reconcile quantum theory with the nature of the

[21] Rigorously defining this path-integral has proven to be difficult, a fact that physicists tend to signal by prefacing its definition with the term 'heuristic'.

macroscopic world, the macroscopic world is less classical than commonly believed: the universe is an interference pattern.

Let's turn now to our second issue: the decomposition of mixed classical states. First recall that in classical mechanics, the space of all states corresponds to the set of probability measures on phase space, and this set does indeed possess the structure of a *simplex*. A simplex of states is distinguished from a convex set of states, in that each element in the interior of a simplex has a *unique* decomposition as a convex combination of extremal boundary elements, i.e., pure states.

In contrast, the space of all states of a quantum system is never a simplex due to the non-unique decomposability of density operators. Hence a mixed quantum state doesn't have a unique decomposition as a convex combination of pure states, and cannot be unambiguously interpreted as representing the relative fraction of various pure states in a population ('ensemble') of systems.

However, Honegger and Rieckers demonstrate that the simplex of classical states can be isomorphically mapped into the total space of quantum states. Let's briefly review how they define this in terms of Weyl algebras.

Start with a real pre-symplectic space (E, σ), which in the case of a field system is infinite-dimensional, possibly a space of solutions to some relativistic wave-equation. The dual E' of this space is the classical phase space. The functions $\Phi^0(f) : E' \to \mathbb{R}$, with $f \in E$, are field observables, defined for $F \in E'$ by $F \mapsto F(f) = \Phi^0(f)[F]$. To obtain a family of bounded functions, these can be exponentiated:

$$W^0(f) = \exp(i\Phi^0(f)) : E' \to \mathbb{C}\,.$$

The closure of the complex linear hull of the $W^0(f)$, $f \in E$, is the commutative Weyl algebra $\mathscr{W}(E, 0)$.

The real-linear mapping $E \ni f \mapsto W^0(f)$ over a pre-symplectic space (E, σ) is called a classical Weyl system. A canonical quantization replaces a classical Weyl system $W^0(f)$ over (E, σ), with a Weyl system $W^\hbar(f)$ over $(E, \hbar\sigma)$. This is a real-linear mapping $E \ni f \mapsto W^\hbar(f)$ into the unitary operators on some Hilbert space, which satisfies the Weyl relations:

$$\begin{aligned} W^\hbar(f)W^\hbar(g) &= \exp(-\tfrac{i}{2}\hbar\sigma(f,g))W^\hbar(f+g)) \\ W^\hbar(f)^* &= W^\hbar(-f) \end{aligned}$$

The non-commutative C*-algebra generated by the Weyl operators is called the Weyl algebra $\mathscr{W}(E, \hbar\sigma)$.

Having fixed a Weyl algebra $\mathscr{W}(E, \hbar\sigma)$, different complex structures j defined on the pre-symplectic space (E, σ), define different semi-norms $||\cdot||_j$ (and inner-products $(\cdot|\cdot)_j$) on the complexified space E. These correspond to different vacuum states ω^j_{vac} on the Weyl algebra, defined by $\langle \omega^j_{vac}, W(f)\rangle = \exp(-\frac{\hbar}{4}||f||_j)$.

Now, suppose we have the pre-symplectic space E, and we have selected a complex structure j, and defined a 1-particle Hilbert space $\mathscr{H}_j$ as the completion of $(E, (\cdot|\cdot)_j)$. Each pure state $e \in \mathscr{H}_j$ defines a character on E:

$$\chi_e(\cdot) = \exp(i\sqrt{2}\,\mathrm{Re}(e|\cdot)_j)\ .$$

Each such character corresponds to a pure state ω_{χ_e} on the commutative Weyl algebra $\mathscr{W}(E,0)$:

$$\langle \omega_{\chi_e}; W^0(f)\rangle = \chi_e(f), \ \ \forall f \in E \ .$$

The state space $S(\mathscr{W}(E,0))$ on the commutative Weyl algebra is closed under convex combinations.

Now, with the selection of the vacuum vector state ω^j_{vac} corresponding to the complex structure j, we have a mapping from the Bauer[22] simplex of states on the commutative Weyl algebra into the total state space:

$$\omega^j_{vac} : S(\mathscr{W}(E,0)) \to S(\mathscr{W}(E,\hbar\sigma)) \ .$$

We can therefore map each ω_{χ_e} into the total state space. Each one of these pure states in the total state space can be regarded as a generalized Glauber vector state, (Honegger and Rieckers, p657), i.e. a coherent state. Moreover, the unique extremal decomposition of each element of $S(\mathscr{W}(E,0))$ is carried over via ω^j_{vac} into the classical states in the total state space.

Hence, Honegger and Rieckers represent the space of all classical coherent states of the radiation field as a Bauer simplex in which "the extremal coherent states are pure and given by Glauber vectors, which in turn are indexed by one-photon wave-functions," (ibid, p703).

In an earlier section we introduced the concept of unpolarized light as a mixed quantum state which admits more than one decomposition. Unpolarized light can, for example, be represented as a mixture of either linearly polarized pure states, or a mixture of circularly polarized pure states; these expressions are mixtures of photon-number eigenstates. Let's spell this out in terms of states on Weyl algebras.

Consider the following four modes $e_i \in \mathscr{H}_j$, each with the same wave-vector $\mathbf{k}$: Let e_x denote the mode with linear polarization along the x-axis, let e_y denote the mode with linear polarization along the y-axis, let e_+ denote the mode with left-circular polarization, and let e_- denote the mode with right-circular polarization.

The four states of different polarization defined above therefore correspond to four characters, $\chi_{e_x}, \chi_{e_y}, \chi_{e_+}, \chi_{e_-}$, which in turn correspond to four pure states $\omega_{\chi_{e_x}}, \omega_{\chi_{e_y}}, \omega_{\chi_{e_+}}, \omega_{\chi_{e_-}}$ on the commutative Weyl algebra.

The state space $S(\mathscr{W}(E,0))$ on the commutative Weyl algebra is closed under convex combinations, hence the following pair of mixed states also belong to $S(\mathscr{W}(E,0))$:

$$M_{xy} = 0.5\omega_{\chi_{e_x}} + 0.5\omega_{\chi_{e_y}} \ ,$$
$$M_{+-} = 0.5\omega_{\chi_{e_+}} + 0.5\omega_{\chi_{e_-}} \ .$$

These two classical mixed states do not equal each other:

$$M_{xy} \neq M_{+-} \ .$$

[22] A Bauer simplex is a simplex with a compact extremal boundary.

Now, with the selection of the vacuum vector state ω^j_{vac} corresponding to the complex structure j, we have a mapping from the Bauer simplex of states on the commutative Weyl algebra into the total state space:

$$\omega^j_{vac} : S(\mathscr{W}(E,0)) \to S(\mathscr{W}(E,\hbar\sigma)) \ .$$

We can therefore map the four polarization states into the total state space, $\omega^j_{vac}(\omega_{\chi_{e_x}}), \omega^j_{vac}(\omega_{\chi_{e_y}}), \omega^j_{vac}(\omega_{\chi_{e_+}}), \omega^j_{vac}(\omega_{\chi_{e_-}})$. To reiterate, each one of these pure states in the total state space can be regarded as a generalized Glauber vector state.

The unique extremal decomposition of each element of $S(\mathscr{W}(E,0))$ is carried over via ω^j_{vac} into the classical states in the total state space. Thus, we have the following pair of mixed states in the total quantum state space:

$$D_{xy} = 0.5\,\omega^j_{vac}(\omega_{\chi_{e_x}}) + 0.5\,\omega^j_{vac}(\omega_{\chi_{e_y}}) \ ,$$
$$D_{+-} = 0.5\,\omega^j_{vac}(\omega_{\chi_{e_+}}) + 0.5\,\omega^j_{vac}(\omega_{\chi_{e_-}}) \ .$$

Once again, these mixed states, now representable as density operators on Fock space, do not equal each other:

$$D_{xy} \neq D_{+-} \ .$$

These states are mixtures of coherent states on Fock space, each of which is indexed by 1-particle states that share the same mode $\mathbf{k}$, but differ in their polarization.

These mixtures need to be carefully distinguished from the following mixed state. Let P_{e_i} denote the projection operator onto the ray spanned by e_i in the 1-particle subspace of Fock space. A state of unpolarized light can be represented by a density operator with the following non-unique decomposition into a convex combination of orthogonal pure states:

$$D = 0.5P_{e_x} + 0.5P_{e_y} = 0.5P_{e_+} + 0.5P_{e_-} \ .$$

This is obviously not a mixture of coherent states, it is a mixture of photon-number eigenstates.

The density operator D on the 1-particle space can be second-quantized into a density operator $d\Gamma(D)$, a mixed state on Fock space, and this mixed state has a non-unique decomposition:

$$d\Gamma(D) = 0.5\,d\Gamma(P_{e_x}) + 0.5\,d\Gamma(P_{e_y}) = 0.5\,d\Gamma(P_{e_+}) + 0.5\,d\Gamma(P_{e_-}) \ .$$

Once again, however, $d\Gamma(D)$ is not a coherent state. In this respect, a state of unpolarized light is very much a quantum rather than a classical state of light.

Note also that much of the visual wavelength light humans perceive *is* polarized. Sunlight, and the light from incandescent bulbs, becomes linearly polarized by virtue of undergoing Rayleigh scattering with atmospheric molecules, and by virtue of reflecting off non-metallic surfaces. As we note in the next subsection,

these are 'coherent scattering' processes. They transform unpolarized light, a mixed state of the radiation field, into linearly polarized light, another mixed state, but one in which the mixture is only a convex combination of coherent states with different wave-vectors, phases and amplitudes. The coherent scattering processes with matter effectively perform a measurement-like interaction on the light, selecting a basis of coherent states, and in particular, decomposing the mixture in the basis of linearly polarized states. So, whilst laser light is a polarized pure classical state of the radiation field, polarized sunlight is a mixed classical state.

Let's turn to our third issue: the notion that a coherent state is a classical state of the radiation field undermines one particular popular notion that the classical world emerges from the quantum in the limit in which the number of particles tends to infinity. For example, Haag appears to link this to Bohr's 'correspondence principle': "Classical electrodynamics is not the quantum theory of a single photon but a correspondence limit of the quantum theory for infinitely many photons," (1996, p6).

To take another example, Sakurai's well-known textbook on 'advanced' quantum theory asserts that classical electromagnetism is reliable when the number of wavelength-λ photons per unit of volume, $(\lambda/2\pi)^3$, is much greater than one.[23] He estimates that for a Chicago FM radio station broadcasting at $(\lambda/2\pi) \approx 48\,\mathrm{cm}$, the number per unit volume at a distance of 5 miles from the antenna is about 10^{17}:

> "The classical limit of the quantum theory of radiation is achieved when the number of photons becomes so large that the occupation number may as well be regarded as a continuous variable. The space-time development of the classical electromagnetic wave approximates the dynamical behavior of trillions of photons," (1967, p36).

On the contrary, if coherent states truly are pure classical states, and if the Bauer simplex obtained by taking convex linear combinations of the coherent states is a set of mixed classical states, then classical states of the radiation field can be associated with a small, finite expected number of photons.

Troup and Perlman (1981) point out that in the limit where the expected number of photons $\langle n\rangle = |\alpha|^2$ in a coherent state $|\alpha\rangle$ tends to infinity, because the relative uncertainty $|\alpha|/|\alpha|^2 = |\alpha|^{-1}$ tends to zero, the distribution in photon number tends towards that of the photon-number eigenstates $|\langle n\rangle\rangle$. However,

[23] Sakurai divides the energy density of a classical field by $\hbar\omega$ to estimate the number of photons per unit volume, and assumes that the classical theory will be reliable when the vacuum fluctuations of the field, of order $\hbar\omega$ per unit volume, are negligible compared to the classical energy density. Sakurai's simple method of estimating N, the number of photons per unit volume, seems to be a monochromatic version of Zeldovich's more general 1966 formula:

$$N = \frac{1}{8\pi}\int \frac{|\tilde{\mathbf{E}}(\mathbf{k},t)|^2 + |\tilde{\mathbf{B}}(\mathbf{k},t)|^2}{\hbar\omega(\mathbf{k})} d^3\mathbf{k}\,.$$

Interpreted literally, Zeldovich's formula neglects the fact that a classical state is not a photon-number eigenstate.

given that coherent states are classical states and photon-number eigenstates are quantum states, this line of argument would only demonstrate that the quantum emerges from the classical as the number of photons tends to infinity, the very opposite of the claim made by Haag and Sakurai.

A classical state of the radiation field is *not* obtained in the limit where there are many, many photons in a quantum state. In fact, if the state of the radiation field is an eigenstate of the photon number operator, then no matter how large the number of photons, (10^{23}, 10^{100}, etc), it will not correspond to a classical state. It is the coherent states, and the mixtures of coherent states, not the photon number eigenstates, which correspond to classical states. Moreover, each coherent state is a superposition of photon number eigenstates. Hence, in the case of the radiation field, the classical world emerges from superpositions. To reconcile the theory with macroscopic phenomena, not only is it unnecessary to eradicate superpositions, but on the contrary they are *indispensable* for the emergence of classical phenomena.

The classical world is also not associated with a particular energy-scale. In the case of Fock space, the classical coherent states are states of arbitrarily high-energy. There is no upper limit on the expectation value of the particle number in a coherent Fock state. Moreover, the quantum states are the states whose energy can be as low as that associated with a single photon, or with the vacuum itself.

There is one further problem with the notion of a coherent state as a classical state, and in fact it is the same problem which besets the notion of the quantum vacuum.

Every inertial (i.e., non-acceleratory) reference frame in Minkowski space-time shares a common vacuum state, a common number operator, and a common decomposition of the solution space to a relativistic differential equation into positive-energy and negative-energy states. Uniformly accelerated observers, however, have their own vacuum state, their own number operator, and their own notion of positive-energy and negative-energy. In the inertial Minkowski vacuum state, a uniformly accelerated particle detector will purportedly detect a thermal flux of particles, (the so-called 'Unruh effect'). These problems are accentuated for quantum field theory in curved space-time, where not only does each local freely-falling reference frame, defined by a geodesic of the curved space-time geometry, possess its own notion of the vacuum, but the vacuum can change from one moment to another within that fixed reference frame.

Now, the family of coherent states $|\alpha\rangle$ includes the vacuum state as the special case where $\alpha = 0$, and it transpires that this entire class of states is mutable under transition to an accelerated reference frame. Bishop and Vourdas (1986) demonstrated that, not only do uniformly accelerated observers have their own notion of a vacuum state, they also have their own notion of coherent states in general. A single-mode coherent state in the reference frame of an inertial observer, becomes a paired-mode state for a uniformly accelerated observer, a particular type of 'squeezed' state.[24]

[24]Tracing over one of the modes leaves the remaining mode in a thermal quantum state.

Along with the photon-number eigenstates, squeezed states are considered to be non-classical states of light: "Squeezed states of light are nonclassical states of light with phase-dependent quantum noise properties which can be less than those of the vacuum state of the electromagnetic field," (Knight 1989, p313).

The coherent states of the radiation field are sharp up to the quantum fluctuations of the vacuum. This is best explained in terms of the 'quadratures' of a wave. As a trigonometrical identity,

$$\begin{aligned}\sin(x+\phi) &= \cos(\phi)\sin(x) + \sin(\phi)\cos(x) \\ &= X_1 \sin(x) + X_2 \cos(x) \ .\end{aligned}$$

This is a sum of the 'in-phase' quadrature component, and the component 90° out of phase with it. When $\phi = 0$, $\sin(\phi) = 0$, $X_1 = 1$ and $X_2 = 0$. In terms of the electric field, one has (Knight, p314):

$$E = E_0(X_1 \sin(\omega t) + X_2 \cos(\omega t)) \ .$$

The corresponding decomposition of the quantized electric field $\widehat{E}$ contains operators $\widehat{X}_1$ and $\widehat{X}_2$, which are non-commuting. $\widehat{X}_1$ represents the amplitude and $\widehat{X}_2$ represents the phase.[25] They satisfy an uncertainty relationship:

$$\Delta\widehat{X}_1 \cdot \Delta\widehat{X}_2 \geq \frac{1}{4} \ .$$

A coherent state of the radiation field is such that $\Delta\widehat{X}_1 \cdot \Delta\widehat{X}_2 = \frac{1}{4}$ and $\Delta\widehat{X}_1 = \Delta\widehat{X}_2 = \frac{1}{2}$. A squeezed state is such that $\Delta\widehat{X}_i < 1/2$ for one of the quadratures, and $\Delta\widehat{X}_j > 1/2$ for the other.

When the electric field fluctuations are plotted over time as an interval either side of the sinusoidally varying mean value, reducing the uncertainty $\Delta\widehat{X}_1$ reduces the spread in amplitude at the expense of increasing the spread in phase, while reducing the uncertainty $\Delta\widehat{X}_2$ reduces the spread in phase at the expense of increasing the spread in amplitude.

A sufficient condition for the non-classicality of an optical state is that the variance of some field operator is *less* than that in the vacuum state, and by virtue of this, squeezed states are deemed to be non-classical (Honegger and Rieckers 2015, Criterion 25.2.8, p659).

Hence, if one cleaves to the notion that a coherent state is a classical state, then the division between the quantum world and the classical world is not reference-frame independent. Classical states in one reference frame can become non-classical states in another.

[25] There are other popular quadrature operators in the literature. For example, one often sees the annihilation operator $\widehat{a}$ decomposed as $\widehat{a} = (\widehat{a} + \widehat{a}^*)/2 + i(\widehat{a} - \widehat{a}^*)/2i = \widehat{X}_1 + i\widehat{X}_2$. With respect to a coherent state $|\alpha\rangle$, recall that $\widehat{a}|\alpha\rangle = |\alpha|e^{i\theta}|\alpha\rangle$. Hence,

$$\langle\alpha, (\widehat{X}_1 + i\widehat{X}_2)\alpha\rangle = \alpha = \langle\widehat{X}_1\rangle + i\langle\widehat{X}_2\rangle = \mathrm{Re}\,\alpha + i\mathrm{Im}\,\alpha = |\alpha|\cos\theta + i|\alpha|\sin\theta \ .$$

7.2 Coherent scattering

Just as there are coherent and non-coherent states of the quantized radiation field, scattering interactions between radiation and matter can be classified as coherent scattering or incoherent scattering. When the incident radiation behaves in the manner of a classical wave, it is said to be coherent scattering. Two such examples are Rayleigh scattering and Mie scattering.

The basic idea behind coherent scattering is that the electric field of the incoming wave exerts forces on the electrical charges of an atom, displacing the centroid of the negative electron cloud in one direction, and the positive nucleus in the opposite direction, then reversing the process. This creates an oscillating electric dipole, which emits electromagnetic radiation of its own. The result is a superposition of the transmitted incident wave, and the scattered wave radiated by the electric dipole.

Coherent scattering processes are elastic scattering interactions between low-energy radiation and bound atomic electrons. The atom recoils to conserve momentum, and the outgoing radiation has almost the same energy as the incoming radiation, but its direction is deflected by a small angle.

Rayleigh scattering occurs when the wavelength is larger than the target particle, and Mie scattering occurs when the wavelength is similar to the target. In the Earth's atmosphere, Rayleigh scattering occurs between visual wavelength radiation and gaseous molecules such as O_2, while Mie scattering occurs with particles of dust or pollen.

When the wavelength is much smaller than the size of the target, then light can be represented as 1-dimensional rays, and the interaction between light and matter can be described with geometrical optics. Hence, geometrical optics is essentially a limit of coherent scattering.

Thomson scattering is another type of coherent, elastic scattering interaction, but in this case it occurs between radiation and an individual atomic electron, or an unbound electron. The radiation energy is almost unchanged, but it can scatter in any direction.

In contrast, Compton scattering is an *incoherent* scattering interaction between radiation and an atomic electron. The incoming radiation is considered to be in a particle-like photon state, and the photon transfers some of its energy to the electron, as well as changing direction. The incoming photon has an energy much greater than the binding energy of atomic electrons, and whilst the total energy-momentum of the photon and electron is conserved, the atom is ionized in the process.

The importance of coherent scattering will become apparent in the next section when we consider the role played by human visual perception in the conception of the macroscopic world.

8 Decoherence

In this section, we will consider the role of the radiation field in 'decoherence', a process purportedly crucial to understanding the relationship between quantum theory and the macroscopic world.

First, however, it might be prudent to highlight the potentially confusing double-use of the term 'coherent' within quantum physics. Particularly in the context of discussions of the measurement problem, a superposition of pure states is sometimes referred to as a 'coherent superposition'. This is a different use of the word, and doesn't mean that the system is in a coherent state, or even that the components of the superposition are coherent states. As Okon and Sudarsky (2016) put it, "it is hard to see what the adjective of 'coherent' adds to the standard notion of a superposition."

The basic idea of decoherence is as follows. When the joint state of a system and measuring apparatus (or environment) is a superposition in the joint Hilbert space, one can trace over the degrees of freedom of the measuring apparatus (or environment) to obtain the reduced state of the system. This reduced state is represented by a density operator ρ_S. The proponents of decoherence try to interpret ρ_S as a mixed state. They claim that decoherence is a process by which the interaction Hamiltonian between system and measuring apparatus (or environment) rapidly diagonalizes the density operator, in a basis determined by the quantity being measured.

"It is argued that ρ_S is, for all practical purposes the tool to use in order to make predictions regarding all possible measurements to be carried out on the system. And since ρ_S is identical to a mixed state, the results of all these possible measurements are going to be identical to those of measurements performed on a mixed state. That is, for all practical purposes the system will behave as a mixture. Decoherence, then, is said to lead to effectively nonunitary dynamics for the system, which explains the absence of interference between the components of the superposition," (Okon and Sudarsky, 2016).

One of the interpretational problems faced by quantum mechanics is to explain why macroscopic systems, composed entirely of interacting quantum systems, do not appear to exist in superposed states. In other words, why is there no apparent interference between macroscopically different states?

The question, however, is posed, explicitly or implicitly, in terms of the states of matter alone, not the states of the radiation field. Interference effects have been apparent in the radiation field, long before the advent of quantum theory, and these interference effects are associated with *classical* states of the radiation field, represented either in Maxwellian electromagnetism, or its predecessor, wave optics. Hence the radiation field potentially plays a unique role in understanding the relationship between the classical and the quantum.

It is crucial to note that because human visual perception of the surrounding spatial world is based upon detecting a pattern of light in the narrow range of wavelengths $400 - 700$ nm, humans perceive a pattern of radiation which has primarily been reflected or scattered from matter by *coherent* scattering processes. This bath of visual wavelength radiation can be thought of as a type of

environment for the material systems on the Earth, but equally one can think of the material systems as providing an environment for the radiation. In accordance with the dictums of 'environmental decoherence', we can hypothesize that the continuous coherent scattering interactions between visual wavelength radiation and matter diagonalizes the reduced state of the radiation field in a basis of coherent states. The polarization of sunlight is one particular manifestation of this.

Let's consider a scattering scenario. Suppose we have a material system in an initial superposition,

$$\psi = \alpha\psi_{\mathbf{x}_1} + \beta\psi_{\mathbf{x}_2} \ ,$$

where $\psi_{\mathbf{x}_1}$ represents a 'coherent' state of mean position $\mathbf{x}_1$, and $\psi_{\mathbf{x}_2}$ represents a 'coherent' state of mean position $\mathbf{x}_2$, and $||\mathbf{x}_2 - \mathbf{x}_1||$ is larger than the dispersion $\Delta\mathbf{x}_i$ of either coherent state.

These coherent states are assumed to be the states of a massive fermion, or an aggregate system which contains massive fermions. We earlier rejected the possibility of coherent fermionic states, but in that context they were Annihilation Operator Coherent States (AOCS). i.e., they were field-theoretic superpositions in Fock space of particle-number eigenstates of the same mode. In this context, the coherent fermionic states are Minimal Uncertainty Coherent States (MUCS). Specifically, they are non-relativistic states which belong to the same family as the coherent states of a simple harmonic oscillator originally identified by Schrödinger in 1926; i.e., they are localized in phase space, in the sense that the product $\Delta x \cdot \Delta p$ is minimised, the dispersion of each quantity remains fixed in time, and the mean values $\langle x \rangle$ and $\langle p \rangle$ evolve according to the classical equations of motion.

Before fully expounding our scattering scenario, a brief digression on the subject of the harmonic oscillator may be helpful to counter potential anxieties about the physical prevalence of such systems. Recall that harmonic oscillator potentials are those with the quadratic form:

$$V(x) = \frac{1}{2}kx^2 \ .$$

In such a potential, the restoring force on a system perturbed from its equilibrium state is linearly proportional to the displacement, $m\ddot{x} = -kx$. In contrast, chemistry is dominated by the Coulomb potential, which has the form:

$$V(x) = -\frac{k}{x} \ .$$

There are two senses, however, in which the harmonic potential has empirical relevance. First, electrostatic shielding in systems with positive and negative charges creates a variety of effective potentials. In the case of diatomic molecules, for example, the vibrations along the internuclear axis can be represented by a harmonic potential. Second, the small perturbations of any system near a local minimum x_0 in its potential can be represented by a harmonic potential, irrespective of the global form of the potential. This is easy to see with

a Taylor series expansion:

$$V(x) = V(x_0) + (x - x_0)\frac{dV}{dx} + \frac{(x - x_0)^2}{2!}\frac{d^2V}{dx^2} + \cdots$$

By definition, at the local minimum the gradient vanishes, hence $dV/dx = 0$. Thus, removing the offset $V(x_0)$, for small x one has:

$$V(x) \approx \frac{1}{2}k(x - x_0)^2 \ .$$

So, given that harmonic potentials are physically prevalent in this sense, the states of bound fermions close to local minima in their effective potentials, can be approximately represented as coherent states.

It's important to note here that these coherent fermion states must be *bound* states, because the wave-packet of a free-particle in non-relativistic quantum mechanics has a notorious tendency to spread. In the non-relativistic quantum mechanics of a harmonic oscillator one typically has an 'external' classical potential, but this potential is ultimately a quantum field itself. A fermionic system is bound by virtue of its interactions with other systems, and those interactions are mediated by gauge bosons. A fermion can only behave as a simple harmonic oscillator if there is a harmonic potential energy function, in which the restoring force is provided by gauge bosons. Hence, a fermion in a coherent state is implicitly a fermion which is constrained by the nature of bosonic fields.[26]

Returning to our scattering scenario, if the system in an initial superposition $\psi = \alpha\psi_{\mathbf{x}_1} + \beta\psi_{\mathbf{x}_2}$, is an aggregate large enough to be macroscopically observable, this type of state is referred to as a 'macroscopic superposition'. It's worthwhile making explicit what such a state would look like.

If we assume that each $\psi_{\mathbf{x}_i}$ is a normalized state of a system of mass $m = \int m|\psi_{\mathbf{x}_i}|^2 d\mathbf{x} = \int \rho \, d\mathbf{x}$, where ρ is the mass density, then each branch of this superposition contains only a fraction of the total mass. i.e., if $\psi_{\mathbf{x}_i}$ has support in a bounded open region Σ_i of volume $\mathrm{Vol}(\Sigma_i)$, then

$$\int_{\Sigma_1} m|\psi|^2 \, d\mathbf{x} = |\alpha|^2 m$$
$$\int_{\Sigma_2} m|\psi|^2 \, d\mathbf{x} = |\beta|^2 m \ .$$

This reduction in the mass density changes the optical depth τ of each volume Σ_i of space occupied by the branches of the wave-function. Let κ_ν denote the opacity at frequency ν. The opacity κ_ν is the sum of the absorption cross-section $\sigma_a(\nu)$ and the scattering cross-section $\sigma_s(\nu)$, per unit mass of the target particles. For an incident flux I_0, the flux I of light transmitted through an object of diameter l at the specified frequency ν is given by:

[26] An aggregate of fermions is automatically a boson-fermion composite. Specifically, it is a system of fermions bound together by gauge bosons. However, this in itself is clearly insufficient to render the system classical. Neutrons, for example, are composed of fermions (quarks) bound by gauge bosons (gluons), yet neutron behaviour is strongly quantum, as illustrated by the wavelike beam splitting and re-combination phenomena of neutron interferometry.

$$\frac{I}{I_0} = e^{-\kappa_\nu \rho l} = e^{-\tau} \ .$$

Now, while transparent objects have an optical depth of $\tau \approx 0$, translucent ones are defined to have $0 < \tau \leq 4$, and opaque objects have $4 < \tau$, (Lynch and Livingston, 2001, p267).

Hence, the greater the number of branches in a macroscopic superposition, and the lower the density associated with each branch, the closer each branch tends towards representing a translucent object. In other words, a macroscopic superposition would be visually identifiable not merely by multiple images of the same object, but by virtue of those multiple images becoming increasing translucent as the number of branches increases.[27]

Let's return to the initial superposition defined above. Suppose that a state of the radiation field interacts ('scatters') off the target material system in a way which is sensitive to $\mathbf{x}_i$. In other words, we have an incoming state ϕ^{in} of the radiation field, so that the initial state of the combined system is:

$$\psi \otimes \phi^{in} \ ,$$

and we have an outgoing state of the radiation field ϕ^{out}, which is such that:

$$\begin{aligned} \psi_{\mathbf{x}_1} \otimes \phi^{in} &\rightarrow \psi_{\mathbf{x}_1} \otimes \phi_1^{out} \\ \psi_{\mathbf{x}_2} \otimes \phi^{in} &\rightarrow \psi_{\mathbf{x}_2} \otimes \phi_2^{out} \ . \end{aligned}$$

ϕ^{in} might be a plane wave of wave number k, ϕ_1^{out} might be a superposition of a plane wave and an outgoing spherical wave centred on $\mathbf{x}_1$, and ϕ_2^{out} might be a superposition of a plane wave and an outgoing spherical wave centred on $\mathbf{x}_2$, (see Prugovecki 1981, Sections 4.2 and 4.3). The mapping above between initial and final states defines the unitary evolution of the interacting system, hence it implicitly specifies the interaction Hamiltonian.

Now, the linearity of unitary evolution entails that:

$$\psi \otimes \phi^{in} \rightarrow \alpha \left(\psi_{\mathbf{x}_1} \otimes \phi_1^{out} \right) + \beta \left(\psi_{\mathbf{x}_2} \otimes \phi_2^{out} \right) .$$

In this event, the state of the radiation field has become 'entangled' with the state of the material system. The reduced density operator of the material system after the interaction has the following matrix elements in the $\psi_{\mathbf{x}_i}$ basis:

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \langle \phi_2^{out}, \phi_1^{out} \rangle \\ \alpha^* \beta \langle \phi_1^{out}, \phi_2^{out} \rangle & |\beta|^2 \end{pmatrix} \ .$$

When $\langle \phi_2^{out}, \phi_1^{out} \rangle \simeq 0$, the off-diagonal 'interference' terms almost vanish due to the individual scattering event. This is the case when $k||\mathbf{x}_2 - \mathbf{x}_1|| \gg 1$. i.e.,

[27] The other spatial senses would also detect macroscopic superpositions. The reduction in density in the region of space occupied by each branch would reduce the resistance to touch, given that the pressure of a solid ultimately derives from: (i) electrostatic Coulomb repulsion between negatively charged electrons; and (ii) the Pauli exclusion principle.

when the wavelength λ of the incoming radiation state is smaller than the spatial separation of the superposed coherent states. In this case, the individual scattering event will induce a reduced state of the material system which generates the same subsequent pattern of measured values as a mixture of the two coherent states.

When $\langle \phi_2^{out}, \phi_1^{out} \rangle \simeq 1$, the individual scattering event does not destroy the interference terms, but does make a contribution towards diagonalizing the reduced density operator. This is the case where $k||\mathbf{x}_2 - \mathbf{x}_1|| \ll 1$. i.e., when the wavelength λ of the incoming radiation state is longer than the spatial separation of the superposed coherent states. In this case, the systems only become weakly entangled, and the reduced state of the material system generates a similar subsequent pattern of measured values as a pure state superposition of the two coherent states. However, repeated scattering events due to a flux of incoming radiation *will* drive the diagonal elements towards zero.[28] On the assumption of such a flux F, expressions can be derived which show exponentially fast decay of the interference terms, (Wallace 2012, p80; Joos 2003, p65-66):

$$\exp(-\Lambda t(|\mathbf{x}_2 - \mathbf{x}_1||)^2) \ ,$$

where

$$\Lambda \sim k^2 F \sigma \ ,$$

with σ denoting the total interaction cross-section.

These expressions apply to decoherence in general, not merely that induced by scattering with the radiation field. For a 10-micron dust particle suspended in the atmosphere, $\Lambda \sim 10^{36}\, m^{-2} s^{-1}$ due to scattering with atmospheric molecules, $10^{21}\, m^{-2} s^{-1}$ due to the flux of sunlight, and $10^{6}\, m^{-2} s^{-1}$ due to the flux of cosmic microwave background radiation. Joos (p67) estimated a decoherence timescale of order 10^{-13} s due to scattering of air molecules off such a dust particle. Hence, the scattering between boson-fermion composites in a gas or fluid is most effective for decoherence. Radiation (or even possibly the gravitational field) will be the prime agent for material systems not immersed in a gas or fluid.

As Rosaler points out, "[the spreading of fermionic wave-packets] will result in branching as the coherent superposition that results from the spreading is decohered by interaction either with the electromagnetic field or with other fermions... the factors that affect the rate of bosonic wave packet spreading are simply those that affect fermionic wave packet spreading, since bosonic packets do not tend to spread of their own accord but only indirectly by virtue of their interaction with fermionic degrees of freedom (for example, a fermionic state consisting of two widely separated wave packets for a single particle will tend to generate a superposition of very different classical electromagnetic field

[28] Nevertheless, the reduced state is a density operator rather than one of the pure coherent states, and as Okon and Sudarsky (2016) point out, one needs to apply the Born rules to the reduced state to obtain a predicted distribution of measured values, the very same Born rules which the decoherence programme seeks to explain.

configurations, where each field configuration can be regarded as being generated by a different one of the quasi-classical fermionic wave packets," (p199-200)

Observers are special types of composite boson-fermion systems, hence the states of observers can become entangled with the rest of the world. The human concept of the macroscopic world is founded largely upon the focusing and absorption of optical wavelength radiation, but also upon other spatial senses such as touch and proprioception. In the visual case the total state space is:

$$\mathscr{H}_{micro} \otimes \mathscr{H}_{macro} \otimes \mathscr{H}_{rad} \otimes \mathscr{H}_{observer} \ .$$

Visual observations are those which occur by virtue of the ambient radiation field becoming entangled with the state of composite boson-fermion systems, and the boson-fermion observer state becoming entangled with the state of the radiation field. In the case of the other spatial senses, the state space is:

$$\mathscr{H}_{micro} \otimes \mathscr{H}_{macro} \otimes \mathscr{H}_{observer} \ ,$$

and the perceptions are those which occur by virtue of the boson-fermion observer state becoming directly entangled with the state of the surrounding macroscopic ('collective') boson-fermion degrees of freedom.

According to decoherence, we don't visually perceive macroscopic superpositions because the macroscopic boson-fermion systems (or degrees of freedom) become entangled with the microscopic systems (or degrees of freedom), and then the visual-wavelength radiation field becomes entangled with the macroscopic systems (or degrees of freedom). Hence, the coherent scattering interactions between visual-wavelength radiation and material systems play a crucial role in the perception that the world is classical at a macroscopic level. But the ultimate import of quantum theory is to reveal that the universe is an evolving interference pattern.

9 Conclusions

The division between the classical and the quantum is less clear-cut in the case of light. Specifically, we have found the following:

- Coherent states are the best candidates for pure classical states of the radiation field.
- Classical states of the radiation field are not emergent from the quantum states.
- Coherent states provide a basis for the photonic Fock space, hence any state of the quantized radiation field can be expressed as a superposition of classical states. Conversely, any coherent state can be expressed as a superposition of photon-number eigenstates, which are quantum states of the radiation field.
- To reconcile the quantum theory of the radiation field with macroscopic phenomena, not only is it unnecessary to eradicate superpositions, but on the contrary they are *indispensable* for the emergence of classical phenomena.
- The vacuum state of any boson field can be decomposed as a superposition of coherent states. This potentially provides an explanation for why there is something classical rather than nothing classical. However, the existence of fermions implies that some form of vacuum decay took place.
- The classical states of the radiation field do not emerge in the limit as the number of photons tends to infinity.
- The class of coherent states is transformed into non-classical 'squeezed' states under a transition to an accelerated reference frame. Hence, the classicality of the radiation field is not reference-frame independent.
- Ordinary sunlight is a mixed state of the radiation field, which is polarized by 'coherent scattering' with atmospheric molecules and non-metallic surfaces. These coherent scattering processes select a basis of coherent states for the decomposition of the mixed state.
- Decoherence is most effective when a massive system is immersed in a gas or fluid of other massive systems. However, human visual perception is dependent upon another level of environmental decoherence, in which visual wavelength radiation becomes entangled with matter by means of coherent scattering processes.
- A fermion in a coherent state is a fermion which is constrained by the nature of bosonic fields. An aggregate of fermions large enough to be macroscopically observable is automatically a boson-fermion composite, a system bound together by gauge bosons. The apparent classicality of the macroscopic world seems ultimately to be dependent upon boson fields.

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